ALGEBRAIC COMBINATORICS

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ABSTRACT We investigate the homology representation of the symmetric group on rank-selected subposets of subword order. We show that the homology module for words of bounded length, over an alphabet of size n, decomposes into a sum of tensor powers of the S_n-irreducible $S_{(n-1,1)}$ indexed by the partition $(n-1,1)$, recovering, as a special case, a theorem of Björner and Stanley for words of length at most k. For arbitrary ranks we show that the homology is an integer combination of positive tensor powers of the reflection representation $S_{(n-1,1)}$, and conjecture that this combination is nonnegative. We uncover a curious duality in homology in the case when one rank is deleted.

We prove that the action on the rank-selected chains of subword order is a nonnegative integer combination of tensor powers of $S_{(n-1,1)}$, and show that its Frobenius characteristic is h-positive and supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 1\}$.

Our most definitive result describes the Frobenius characteristic of the homology for an arbitrary set of ranks, plus or minus one copy of the Schur function $s_{(n-1,1)}$, as an integer combination of the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 2\}$. We conjecture that this combination is nonnegative, establishing this fact for particular cases.

1. INTRODUCTION

Let A^* denote the free monoid of words of finite length in an alphabet A. Subword order is defined on A^* by setting $u \leqslant v$ if u is a subword of v, that is, the word u is obtained by deleting letters of the word v. This makes $(A^*, \leqslant)$ into a graded poset with rank function given by the length $|w|$ of a word w, the number of letters in w. The topology of this poset was first studied by Farmer (1979) and then by Björner, who showed in [5, Theorem 3] that any interval of this poset admits a dual CL-shelling. The intervals are thus homotopy Cohen–Macaulay, as well as all rank-selected subposets obtained by considering only words whose rank belongs to a finite set S [2, Theorem 4.1], [7, Theorem 8.1]. Suppose now that the alphabet A is finite, of cardinality n. The symmetric group S_n acts on A, and thus on A^*. To avoid trivialities we will assume $n \geqslant 2$.

In this paper we describe the homology representation of intervals $[r, k]$ of consecutive ranks in A^*, as well as some other rank-selected subposets, using the Whitney homology technique and other methods developed in [21]. All homology in this paper is taken over the field of complex numbers. We refer the reader to [17] for general facts about rank-selection. We show that the unique nonvanishing homology of the

rank-selected subposet $A^*_{[r,k]}$ decomposes as a direct sum of copies of r consecutive tensor powers of the reflection representation of S_n, that is, the irreducible representation $S_{(n-1,1)}$ indexed by the partition $(n-1,1)$. Theorem 3.7 on consecutive ranks generalises a theorem in [5] (conjectured by Björner and proved by Stanley) on the homology representation of the poset of all words of length at most k. We establish similar results for the Whitney and dual Whitney homology modules. Both turn out to be permutation modules in each degree, with pleasing orbit stabilisers. Theorem 4.2 establishes the nonnegativity property with respect to tensor powers of $S_{(n-1,1)}$ for the case when one rank is deleted from the interval $[1,k]$, and leads to a curious homology isomorphism (Proposition 4.3), suggesting a homotopy equivalence between the simplicial complexes associated to the rank sets $[1,k]\backslash\{r\}$ and $[1,k]\backslash\{k-r\}$, for fixed $r, 1 \leqslant r \leqslant k-1$. Finally Theorem 5.7 establishes that the homology is a nonnegative sum of tensor powers of $S_{(n-1,1)}$ for rank-sets of size 2.

More generally, we show in Theorem 5.2 that for any nonempty subset S of ranks $[1,k]$, the homology representation of S_n may be written as an integer combination of positive tensor powers of the reflection representation. We propose the following conjecture, which is supported by Theorems 3.7, 4.2 and 5.7:

CONJECTURE 1.1. *Let A be an alphabet of size $n \geqslant 2$. Then the S_n-homology module of any finite nonempty rank-selected subposet of subword order on A^* is a* nonnegative *integer combination of positive tensor powers of the irreducible indexed by the partition $(n-1,1)$.*

These considerations lead us to examine the tensor powers of the reflection representation (see Section 6), and the question of how many tensor powers are linearly independent characters. In answering these questions, we are led to a decomposition (Theorem 7.7) showing that the kth tensor power of $S_{(n-1,1)}$ plus or minus one copy of $S_{(n-1,1)}$, has Frobenius characteristic equal to a nonnegative integer combination of the homogeneous symmetric functions $\{h_{(n-r,1^r)} : r \geqslant 2\}$. It is "almost" an h-positive permutation module. (In general the homology itself is *not* a permutation module.) Inspired by this phenomenon, we prove, in Theorem 7.1, that in fact for all rank subsets T, the homology module $\tilde{H}(T)$ has the property that $\tilde{H}(T) + (-1)^{|T|}S_{(n-1,1)}$ has Frobenius characteristic equal to an *integer* combination of the homogeneous symmetric functions $\{h_{(n-r,1^r)} : r \geqslant 2\}$. Theorem 7.3 establishes the truth of the following conjecture for the homology of several subsets of ranks.

CONJECTURE 1.2. *Let A be an alphabet of size $n \geqslant 2$. Then the S_n-homology module of any finite nonempty rank-selected subposet of subword order on A^*, plus or minus one copy of the reflection representation of S_n, is a permutation module. In fact its Frobenius characteristic is h-positive and supported on the set $T_2(n) = \{h_\lambda : \lambda = (n-r,1^r), r \geqslant 2\}$.*

We give a simple criterion for when Conjecture 1.1 will imply Conjecture 1.2 in Lemma 7.9.

The main results of this paper are summarised below. Let A be an alphabet of size $n \geqslant 2$, and $T \subseteq [1,k]$ a subset of ranks.

THEOREM 1.3. *The S_n-module induced by the action of S_n on the maximal chains of the rank-selected subposet of A^* of words with lengths in T, is a* nonnegative *integer combination of tensor powers of the reflection representation $S_{(n-1,1)}$. If $|T| \geqslant 1$, this module has h-positive Frobenius characteristic supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r,1^r), r \geqslant 1\}$.*

THEOREM 1.4. *The homology module $\tilde{H}(T)$ of words with lengths in T is an integer combination of positive tensor powers of the reflection representation $S_{(n-1,1)}$, with the*

property that $\tilde{H}(T) + (-1)^{|T|} S_{(n-1,1)}$ has Frobenius characteristic equal to an integer combination of the homogeneous symmetric functions $\{h_{(n-r,1^r)} : r \geqslant 2\}$.

Both integer combinations are nonnegative *when T is one of the following rank sets:* (1) $[r,k], k \geqslant r \geqslant 1$;   (2) $[1,k]\backslash\{r\}, k \geqslant r \geqslant 1$;   (3) $\{1 \leqslant s_1 < s_2 \leqslant k\}$.

By standard symmetric function theory, any S_n-module is a (virtual) sum of permutation modules whose point-stabilisers are Young subgroups. Thus a salient feature of Theorems 1.3 and 1.4 is that only Young subgroups indexed by hooks appear.

2. Subword order

The subword order poset A^* has a unique least element at rank 0, namely the empty word $\varnothing$ of length zero. In this section we collect the main facts on subword order from [5] that we will need. For general facts about posets, Möbius functions, etc. we refer the reader to [18].

DEFINITION 2.1 ([9]). *A word α in A^* is* normal *if no two consecutive letters of α are equal.*

For example, *aabbccaabbcc* is not normal, while *abcabc* is normal. Normal words are also called *Smirnov* words in recent literature. Observe that the number of normal words of length i is $n(n-1)^{i-1}$.

THEOREM 2.2 (Farmer [9]).

(1) *Let α be any word in A^*. Then the Möbius function of subword order satisfies*
$$\mu(\hat{0}, \alpha) = \begin{cases} (-1)^{|\alpha|}, & \text{if } \alpha \text{ is a normal word,} \\ 0, & \text{otherwise.} \end{cases}$$

(2) *(See also [22].) Let $|A| = n$ and let $A^*_{n,k}$ denote the subposet of A^* consisting of the first k nonzero ranks and the empty word, i.e. of words of length at most k, with an artificially appended top element $\hat{1}$. Then*

(1)
$$\mu(A^*_{n,k}) = \mu(\hat{0}, \hat{1}) = (-1)^{k-1}(n-1)^k.$$

(3) *[9, Theorem 5 and preceding Remark] $A^*_{n,k}$ has the homology of a wedge of $(n-1)^k$ spheres of dimension $(k-1)$.*

Björner generalised Part (1) above to give a simple formula for the Möbius function of an arbitrary interval (β, α), as follows.

DEFINITION 2.3 ([5]). *Given a word $\alpha = a_1 a_2 \ldots a_n$ in A^*, its* repetition set *is $R(\alpha) = \{i : a_{i-1} = a_i\}$. An* embedding *of β in α is a sequence $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $\beta = a_{i_1} a_{i_2} \ldots a_{i_k}$. It is called a* normal embedding *if in addition $R(\alpha) \subseteq \{i_1, i_2, \ldots, i_k\}$.*

Denote by $\binom{\alpha}{\beta}$ the number of embeddings of β in α, and by $\binom{\alpha}{\beta}_n$ the number of normal embeddings of β in α.

THEOREM 2.4 ([5, Theorem 1]). *For all $\alpha, \beta \in A^*$,*
$$\mu(\beta, \alpha) = (-1)^{|\alpha|-|\beta|} \binom{\alpha}{\beta}_n.$$

Observing that the word α is normal if and only if its repetition set $R(\alpha)$ is empty, one sees that this generalises Farmer's formula for $\mu(\hat{0}, \alpha)$.

Recall that the zeta function [18] of a poset is defined by $\zeta(\beta, \alpha) = 1$ if $\beta \leqslant \alpha$, and equals zero otherwise.

THEOREM 2.5 ([5]). *Let A be an alphabet of size n, and β a word in A^* of length k. The following generating functions hold:*

(1) [5, Theorem 2 (i)] *For the Möbius function of subword order:*

$$\sum_{\alpha \in A^*} \mu(\beta, \alpha) t^{|\alpha|} = \frac{t^k(1-t)}{(1+(n-1)t)^{k+1}}.$$

(2) [5, 3. Remark.] *The number of words of length p in the interval $[\beta, \infty]$ depends only on the length k of β, and equals*

$$\sum_{i=0}^{p-k} \binom{p}{i}(n-1)^i.$$

(3) [5, 3. Remark.(i)] *For the zeta function of subword order:*

$$\sum_{\alpha \in A^*} \zeta(\beta, \alpha) t^{|\alpha|} = \frac{t^k}{(1-nt)(1-(n-1)t)^k}.$$

Farmer's result on the homology of $A^*_{n,k}$ was strengthened by Björner, who showed the following (see [2], [7], and also [23] for a survey of lexicographic shellability):

Theorem 2.6 (Björner [5, Theorem 3, Corollary 2]). *Every interval (β, α) in the subword order poset A^* is dual CL-shellable, and hence homotopy Cohen–Macaulay. In particular, for a finite alphabet A, the poset $A^*_{n,k}$ of nonempty words of length at most k, which may be viewed as the result of rank-selection from an appropriate interval of A^*, is also dual CL-shellable and hence also homotopy Cohen–Macaulay.*

We point out three details about Farmer's original paper:
   (1) The order used in the present paper is what Farmer calls the *embedding* order (see [9, p. 609]). Farmer's "subword order" differs from ours and [5].
   (2) All homology in Farmer's paper is ordinary homology, as opposed to reduced homology in the present paper and [17]. In keeping with his definition of a graded poset, for the rank function d of subword order, Farmer defines $d(w) = j - 1$ if w is a word of length j; in this paper we use the length of the word as its rank.
   (3) In particular, his definition of the k-skeleton X^k of a poset of words X corresponds to our A^*_{k+1}, i.e. to taking the words of length at most $(k+1)$.

3. Rank-selection in A^*

In this section we will assume the alphabet A is finite of size n.

We follow the standard convention as in [17], [18]: By the homology of a poset P with greatest element $\hat{1}$ and least element $\hat{0}$, we mean the reduced homology $\tilde{H}(P)$ of the simplicial complex whose faces are the chains of $P \setminus \{\hat{0}, \hat{1}\}$. In order to determine the homology of rank-selected subposets of $A^*_{n,k}$, we will use the techniques developed in [21]. For an elementary treatment of these and more general methods, see [20].

Whitney homology was originally defined by Baclawski [1]. Björner showed [3] that the ith Whitney homology of a graded Cohen–Macaulay poset P with least element $\hat{0}$ is given by the isomorphism

$$(2) \qquad WH_i(P) \simeq \bigoplus_{x:\mathrm{rank}(x)=i} \tilde{H}_{i-2}(\hat{0}, x).$$

Note that if P has a top element $\hat{1}$, then the top Whitney homology coincides with the top homology of P.

If G is a group acting on the Cohen–Macaulay poset P, then $WH_i(P)$ is also a G-module. The present author observed that the isomorphism (2) is in fact group-equivariant, and also established the equivariant acyclicity of Whitney homology

(see [21]). Thus (2) becomes an effective tool for computing both the *G*-module structure of Whitney homology as well as the homology of the full poset, as an equivariant analogue of the inherently recursive structure in the Möbius function. This technique was then exploited in [21] and [20] to determine group actions on the homology of posets.

It is also computationally useful to consider the *dual* Whitney homology of the Cohen–Macaulay poset *P* when *P* has a top element $\hat{1}$, that is, the Whitney homology of the dual poset P^*, which we denote by $WH^*(P)$. Note that we now have an equivariant isomorphism

$$(3) \qquad WH_i^*(P) \simeq \bigoplus_{x:\text{rank}(x)=r-i} \tilde{H}_{i-2}(x, \hat{1}), 0 \leqslant i \leqslant r.$$

Here *r* is the length of the longest chain from $\hat{0}$ to $\hat{1}$.

See [21] and [20] for a more general version of the following theorem (for arbitrary posets), and also [23] for additional background on Whitney homology.

Theorem 3.1 ([21, Lemma 1.1, Theorem 1.2, Proposition 1.9]). *Let P be a graded Cohen–Macaulay poset of rank r carrying an action of a group G. Then the unique nonvanishing top homology of P coincides with the top Whitney homology module $WH_r(P)$, and as a G-module, can be computed as an alternating sum of Whitney homology modules:*

$$(4) \qquad \tilde{H}_{r-2}(P) \simeq \bigoplus_{i=0}^{r-1} (-1)^i WH_{r-1-i}(P).$$

In particular, if $P(\underline{k})$ denotes the subposet consisting of the first k nonzero ranks, with a bottom and top element attached, then one has the G-module decomposition

$$(5) \qquad \tilde{H}_{k-2}(P(\underline{k-1})) \oplus \tilde{H}_{k-1}(P(\underline{k})) \simeq WH_k(P), r \geqslant k \geqslant 1.$$

Note that $WH_0(P)$ is the trivial G-module, while $WH_r(P)$ gives the reduced top homology of the poset P.

Richard Stanley proved Björner's conjecture that

Theorem 3.2 ([5, Theorem 4]). *The S_n-action on the unique nonvanishing homology $\tilde{H}_{k-1}(A_{n,k}^*, \mathbb{C})$ is the module given by the kth tensor power of the irreducible representation indexed by the partition $(n-1, 1)$.*

In the following theorem, we formalise Stanley's insight into subword order, as used in the proof of the above theorem. For more background on the Hopf trace formula and its use in poset homology, see [20]. Recall that the *Lefschetz module* of a poset *P* is the alternating sum (by degree) of the homology modules of (the order complex) of *P*.

Denote by S_λ the irreducible representation of the symmetric group S_n indexed by the partition λ of *n*, and write $S_\lambda^{\otimes i}$ for the *i*th tensor power of the module S_λ.

Theorem 3.3. *Let $\{P_n\}$ be any sequence of finite posets each carrying an action of the symmetric group S_n, such that*

(1) *For any $g \in S_n$, the fixed-point subposet P_n^g is isomorphic to the poset $P_{\text{fix}(g)}$, where $\text{fix}(g)$ is the number of fixed points of g as a permutation of S_n, and*

(2) *the Möbius number $\mu(P_n)$ is a polynomial in $(n-1)$, say $\sum_{i \geqslant 0} b_i(n-1)^i$.*

Then the Lefschetz module of P_n decomposes as a sum of ith tensor powers of the irreducible indexed by the partition $(n-1, 1)$, with coefficient equal to $b_i, i \geqslant 0$. (Note that the 0th tensor power corresponds to the trivial S_n-module $S_{(n)}$.) In particular, the S_n-module structure of the Lefschetz module of P_n is completely determined by its Möbius number.

Now assume P_n is Cohen–Macaulay, as well as all the fixed-point subposets P_n^g. If for all $k \geqslant 0$, the Betti number of $WH_k(P_n)$ is a polynomial in $(n-1)$, then this polynomial determines the trace of $g \in S_n$ on the kth Whitney homology of P_n. The representation of S_n on $WH_k(P_n)$ is therefore a linear combination of tensor powers of the irreducible $S_{(n-1,1)}$.

Proof. This is clear since

(1) ([17, 18, 20]) the Lefschetz module of P_n has (virtual) degree $\mu(P_n)$, the Euler characteristic of the order complex of P_n;

(2) ([18, 20]) by the Hopf trace formula, the trace of an element $g \in S_n$ on this Lefschetz module is the Möbius number $\mu(P_n^g)$ of the fixed-point poset P_n^g, since it is the Euler characteristic of the order complex of P_n^g;

(3) by hypothesis, $\mu(P_n^g) = \mu(P_{\mathrm{fix}(g)}) = \sum_i b_i(\mathrm{fix}(g)-1)^i$, and finally

(4) the trace of g on the irreducible S_n-module indexed by $(n-1,1)$ is $\mathrm{fix}(g)-1$.

Similar conclusions hold for Whitney homology in the case when the posets are Cohen–Macaulay. The key observation here is that from Björner's formulation Equation (2), it follows that the Whitney homology of the fixed-point subposet P_n^g coincides with the Whitney homology of $P_{\mathrm{fix}(g)}$. $\qquad\square$

Our motivating example for the poset P_n satisfying the conditions of Theorem 3.3 is clearly subword order A^* when $|A| = n$. More generally, fix an integer $k \geqslant 1$, and let S be any subset of the ranks $[1, k]$. Then the rank-selected subposet $A_{n,k}^*(S)$ of A^* consisting of elements with ranks belonging to S also satisfies the conditions of Theorem 3.3. When $S = [1, k]$ we denote this rank-selected subposet $A_{[1,k]}^*$ simply by $A_{n,k}^*$.

Using the generating function for the Möbius function of A^* given in Theorem 2.5, Theorem 3.5 below computes all but the top Whitney homology S_n-modules for subword order. The proof requires a key formula, which we derive from the generating function for the Möbius function of A^* given in Theorem 2.5. We isolate this computation in the following lemma.

Lemma 3.4. *Let β be any element of $A_{n,k}^*\backslash\{\hat{1}\}$, where the alphabet A has cardinality n. Then*

$$\mu(\beta, \hat{1})_{A_{n,k}^*}(-1)^{k+1-|\beta|} = \binom{k}{|\beta|}(n-1)^{k-|\beta|}.$$

In particular this Möbius number depends only on the rank (length) of the word β.

Proof. For convenience let $|\beta| = i$. We have, using the defining recurrence for the Möbius function and the generating function in (1) of Theorem 2.5,

$$\mu(\beta, \hat{1})_{A_{n,k}^*}(-1)^{k+1-|\beta|} = (-1)^{k+1-i}(-1)\sum_{\substack{\alpha \in A_{n,k}^* \\ \alpha < \hat{1}}} \mu(\beta, \alpha)$$

$$= (-1)^{k-i}\sum_{j=i}^{k}\sum_{\substack{\alpha \in A_{n,k}^* \\ |\alpha|=j}} \mu(\beta, \alpha)$$

$$= (-1)^{k-i}\sum_{j=i}^{k}[t^j](1-t)t^i(1+t(n-1))^{-(i+1)}$$

$$= (-1)^{k-i}\sum_{j=i}^{k}[t^{j-i}](1-t)(1+t(n-1))^{-(i+1)}.$$

Setting $u = j - i$, this in turn equals

$$(-1)^{k-i} \sum_{u=0}^{k-i} [t^u](1-t)(1+t(n-1))^{-(i+1)} = (-1)^{k-i}[t^{k-i}](1+t(n-1))^{-(i+1)},$$

(since for any power series $f(t)$, one has $\sum_{j=0}^{m}[t^j](1-t)f(t) = [t^m]f(t)$),

$$= (-1)^{k-i} \binom{-(i+1)}{k-i}(n-1)^{k-i}$$

$$= \binom{i+1+k-i-1}{k-i}(n-1)^{k-i}.$$

The last line follows since $\binom{-m}{j} = (-1)^j \binom{m+j-1}{j}$, thereby completing the proof. $\square$

THEOREM 3.5. *Consider the subword order poset $A_{n,k}^*$, with $|A| = n$. As S_n-modules, the Whitney homology $WH(A_{n,k}^*)$ and the dual Whitney homology $WH^*(A_{n,k}^*)$, for $1 \leqslant i \leqslant k$, are as follows. Note that $WH_0(A_{n,k}^*) = S_{(n)} = WH_{k+1}^*(A_{n,k}^*)$ (the trivial S_n-module).*

$$(6) \qquad WH_i(A_{n,k}^*) = S_{(n-1,1)}^{\otimes i} \oplus S_{(n-1,1)}^{\otimes(i-1)};$$

$$(7) \qquad WH_{k+1-i}^*(A_{n,k}^*) = \binom{k}{i} S_{(n-1,1)}^{\otimes(k-i)} \otimes (S_{(n-1,1)} \oplus S_{(n)})^{\otimes i}$$

$$(8) \qquad = \bigoplus_{j=0}^{i} \binom{k}{i}\binom{i}{j} S_{(n-1,1)}^{\otimes j+(k-i)}.$$

Proof. Equation (6) is immediate from Theorems 3.1 and 3.2.

For fixed k, we will show that the Betti number of the kth dual Whitney homology is a polynomial in $(n-1)$ with nonnegative coefficients. By Theorem 3.3, to compute the action of S_n, it is enough to carry out the appropriate Möbius number (in effect, Betti number) computations.

For the dual Whitney homology, for $0 \leqslant i \leqslant k$ we have

$$WH_{k+1-i}^*(A_{n,k}^*) = \bigoplus_{x:|x|=i} \tilde{H}(x, \hat{1})_{A_{n,k}^*}.$$

Computing Betti numbers, and using Lemma 3.4, we have that the dimension of the dual Whitney homology module equals

$$\sum_{\substack{x \text{ any word} \\ |x|=i}} (-1)^{k+1-i}\mu(x, \hat{1})_{A_{n,k}^*} = \sum_{\substack{x \text{ any word} \\ |x|=i}} \binom{k}{i}(n-1)^{k-i} = n^i \binom{k}{i}(n-1)^{k-i}.$$

This expression translates into the one in the statement of the proposition, since the trace of g on $S_{(n-1,1)} \oplus S_{(n)}$ is the number of fixed points of g. The second expression is obtained from the binomial expansion of n^i into powers of $(n-1)$. $\square$

COROLLARY 3.6. *The top homology of $A_{n,k}^*$ as an S_n-module is also given by the alternating sums*

$$\sum_{i=0}^{k}(-1)^{k-i}\left(S_{(n-1,1)} \oplus S_{(n)}^{\otimes i} \otimes \binom{k}{i} S_{(n-1,1)}^{\otimes(k-i)}\right) = \sum_{i=0}^{k}(-1)^{k-i} \bigoplus_{j=0}^{i} \binom{k}{i}\binom{i}{j} S_{(n-1,1)}^{\otimes j+(k-i)},$$

and thus both are equal to $S_{(n-1,1)}^{\otimes k}$.

Proof. The two expressions are simply the alternating sums of dual Whitney homology modules in (7). They equal the top homology module by Theorem 3.1. $\square$

We can now prove the main result of this section, which generalises Theorem 3.2 to the rank-set $[r, k]$ consisting of the interval of consecutive ranks $r, r + 1, \ldots, k$. To do this, we must rewrite the partial alternating sums of terms appearing in the dual Whitney homology (7) as a nonnegative linear combination rather than a signed sum. The poset of words in an alphabet of size n, with lengths bounded above by k and below by r, has homology as follows.

THEOREM 3.7. *Fix $k \geqslant 1$ and let S be the interval of consecutive ranks $[r, k]$ for $1 \leqslant r \leqslant k$. Then the rank-selected subposet $A^*_{n,k}(S)$ has unique nonvanishing homology in degree $k - r$, and the S_n-homology representation on $\tilde{H}_{k-r}(A^*_{n,k}(S))$ is given by the decomposition*

$$(9) \qquad \bigoplus_{i=1+k-r}^{k} b_i \, S^{\otimes i}_{(n-1,1)}, \ \text{where } b_i = \binom{k}{i}\binom{i-1}{k-r}, i = 1 + k - r, \ldots, k.$$

Proof. For brevity we will simply write $\tilde{H}([i, k])$ for the homology of the subposet $A^*_{n,k}(S)$ when $S = [i, k]$. Shellability implies that the rank-selected subposet $A^*_{n,k}(S)$ has unique nonvanishing homology in degree $k - i$.

Recall again from Theorem 3.3 that it suffices to work with the Betti numbers, for which (5) in Theorem 3.1, in conjunction with Theorem 3.5, gives the following recurrence for $1 \leqslant i \leqslant k - 1$:

$$(10) \quad \dim \tilde{H}([i, k]) \oplus \dim \tilde{H}([i + 1, k]) = \dim WH^*_{k+1-i}(A^*_{n,k}) = n^i \binom{k}{i}(n - 1)^{k-i}.$$

We will prove the Betti number version of (9) by induction on i. Note that the result is true for $i = 1$, since in that case the formula in (9) gives simply $S^{\otimes k}_{(n-1,1)}$, with Betti number $(n - 1)^k$, in agreement with Theorem 2.2.

When $r = k$, the formula (9) reduces to $\sum_{i=1}^{k} \binom{k}{i}\binom{i-1}{0}(n-1)^i$, which equals $n^k - 1$. This is easily seen to be the correct Möbius number (up to sign) since we then have a single rank consisting of the n^k words of length k. Also observe that when $i = k - 1$, the recurrence (10) gives

$$\dim \tilde{H}([k - 1, k]) = n^{k-1}k(n - 1) - n^k + 1 = (k - 1)n^k - kn^{k-1} + 1.$$

Let $i = 1$. The recurrence (10) gives

$$\dim \tilde{H}([2, k]) = n\binom{k}{1}(n - 1)^{k-1} - \dim \tilde{H}([1, k])$$

$$= kn(n - 1)^{k-1} - (n - 1)^k = (k - 1)(n - 1)^k + k(n - 1)^{k-1},$$

and this coincides with (9) for $r = 2$.

Assume that (9) holds for the rank-set $S = [r, k]$. We will show that it must hold for $S = [r+1, k]$. By hypothesis we have $\dim \tilde{H}([r, k]) = \sum_{j=1+(k-r)}^{k} \binom{k}{j}\binom{j-1}{k-r}(n-1)^j$, and hence the recurrence (10) gives, for $\dim \tilde{H}([r + 1, k])$, the expression

$$\binom{k}{r}(n - 1)^{k-r}n^r - \sum_{j=1+(k-r)}^{k} \binom{k}{j}\binom{j-1}{k-r}(n - 1)^j.$$

Expanding n^r in powers of $(n - 1)$, we obtain

$$\binom{k}{r}(n - 1)^{k-r}\sum_{i=0}^{r}\binom{r}{i}(n - 1)^i - \sum_{j=1+(k-r)}^{k} \binom{k}{j}\binom{j-1}{k-r}(n - 1)^j.$$

The coefficient of $(n-1)^{k-r}$ is clearly $\binom{k}{r} = \binom{k}{k-r}\binom{r}{0}$, in agreement with (9). For $j = 1 + (k-r), \ldots, r + (k-r)$, the term $(n-1)^j$ appears with coefficient c_j where

$$
\begin{aligned}
c_j &= \binom{k}{r}\binom{r}{j-k+r} - \binom{k}{j}\binom{j-1}{k-r} \\
&= \binom{k}{j}\left(\frac{j!(k-j)!}{r!(k-r)!}\frac{r!}{(j-k+r)!(k-j)!} - \binom{j-1}{k-r}\right) \\
&= \binom{k}{j}\left(\binom{j}{k-r} - \binom{j-1}{k-r}\right) = \binom{k}{j}\binom{j-1}{k-r-1},
\end{aligned}
$$

which is precisely as predicted by (9) for $S = [r+1, k]$. This finishes the inductive step, and hence the proof. $\qquad\square$

This proof establishes the following combinatorial identity, which will be instrumental in the proof of Theorem 7.3 later in the paper.

COROLLARY 3.8.

$$
\sum_{i=0}^{k+1-r} (-1)^i \dim WH^*_{k+1-(r+i)}(A^*_{n,k}) = \sum_{i=0}^{k-r}(-1)^i\binom{k}{r+i}n^{r+i}(n-1)^{k-(r+i)} + (-1)^{k+1-r}
$$

$$
= \sum_{i=1+k-r}^{k}\binom{k}{i}\binom{i-1}{k-r}(n-1)^i.
$$

4. DELETING ONE RANK FROM $A^*_{n,k}$: A CURIOUS ISOMORPHISM OF HOMOLOGY

In this section we will determine the homology representation of the rank-selected subposet $A^*_{n,k}(S)$ of $A^*_{n,k}$ when S is obtained by deleting one rank from the interval $[1, k]$. In this special case the computation will reveal a curious duality in homology.

Again we use a method developed in [21] which is particularly useful for Lefschetz homology computations when the deleted set is an antichain. The version below is the special case when one rank is deleted.

THEOREM 4.1 ([21, Theorem 1.10], [20]). *Let P be a Cohen–Macaulay poset of rank r, G a group of automorphisms of P and let Q be a subposet obtained by deleting a rank-set T consisting of one rank in P. Thus Q is also G-invariant, and Q is graded and has homology concentrated in the highest degree $\mathrm{rank}(Q) - 2$. Then one has the G-equivariant decomposition*

$$
(-1)^{r-\mathrm{rank}(Q)}\tilde{H}(Q) - \tilde{H}_{r-2}(P) = \bigoplus_{x \in T/G} (-1) \cdot (\tilde{H}(\hat{0}, x)_P \otimes \tilde{H}(x, \hat{1})_P) \uparrow^G_{\mathrm{stab}(x)},
$$

where the sum runs over one element $x \in T$ in each orbit of G.
Here $\mathrm{stab}(x)$ denotes the stabiliser subgroup of G which fixes the element x.

We apply this theorem to the poset $A^*_{n,k}$ and the rank-set $S = [1, k] \backslash \{r\}$, removing all words of length r, for a fixed r in $[1, k]$.

THEOREM 4.2. *As an S_n-module, we have*

$$
\tilde{H}_{k-2}(A^*_{n,k}(S)) \simeq \left[\binom{k}{r} - 1\right] S^{\otimes k}_{(n-1,1)} \oplus \binom{k}{r} S^{\otimes k-1}_{(n-1,1)}.
$$

Proof. We invoke Theorem 3.3 by fixing a rank-set S and considering the family of posets $P_n = A^*_{n,S} = A^*_{n,k}(S)$, where $n = |A|$. Once again we need only compute

Möbius numbers in Theorem 4.1. Writing simply $\mu(P)$ for the Möbius number of the poset P, the Betti number identity given by the theorem is

$$- (-1)^{k-2}\mu(A^*_{n,k}(S)) - (-1)^{k-1}\mu(A^*_{n,k})$$
$$= (-1) \cdot \sum_{x:|x|=r} (-1)^r \mu(\hat{0}, x)_{A^*_{n,k}} \cdot (-1)^{k+1-r}\mu(x, \hat{1})_{A^*_{n,k}},$$

or equivalently, clearing signs,

$$\mu(A^*_{n,k}(S)) - \mu(A^*_{n,k}) = (-1) \cdot \sum_{x:|x|=r} \mu(\hat{0}, x)_{A^*_{n,k}} \cdot \mu(x, \hat{1})_{A^*_{n,k}}.$$

The summand corresponding to a word x of length r in the right-hand side of this equation is nonzero only if x is a normal word, by Theorem 2.2. We therefore obtain, using Lemma 3.4,

$$\mu(A^*_{n,k}(S)) - \mu(A^*_{n,k}) = (-1) \cdot (-1)^r n(n-1)^{r-1}\mu(x_0, \hat{1})_{A^*_{n,k}}$$
$$= (-1) \cdot (-1)^r n(n-1)^{r-1}(-1)^{k-r+1}(n-1)^{k-r}\binom{k}{r}$$
$$= (-1)^k n(n-1)^{k-1}\binom{k}{r},$$

for any fixed normal word x_0 of length r.

Hence

$$(-1)^k \mu(A^*_{n,k}(S)) = (-1)^k \mu(A^*_{n,k}) + n(n-1)^{k-1}\binom{k}{r}$$
$$= -(n-1)^k + \binom{k}{r}n(n-1)^{k-1}$$
$$= \left[\binom{k}{r} - 1\right](n-1)^k + \binom{k}{r}(n-1)^{k-1}.$$

Since $A^*_{n,k}(S)$ has rank k, this is precisely the Betti number version of the statement of the theorem, thereby completing the proof. $\square$

An immediate and intriguing corollary is the following.

PROPOSITION 4.3. *Let $|A| = n$. Fix a rank $r \in [1, k-1]$. Then the homology modules of the subposets $A^*_{n,k}([1,k]\setminus\{r\})$ and $A^*_{n,k}([1,k]\setminus\{k-r\})$ are S_n-isomorphic.*

It would be interesting to explain this isomorphism topologically. More precisely:

QUESTION 4.4. *Is there a combinatorial map giving an S_n-homotopy equivalence between the simplicial complexes associated to the subposets $A^*_{n,k}([1,k]\setminus\{r\})$ and $A^*_{n,k}([1,k]\setminus\{k-r\})$?*

5. THE ACTION ON CHAINS, AND ARBITRARY RANK-SELECTED HOMOLOGY

Assume $|A| = n$. For a subset $S \subseteq [1, k]$, denote by $\alpha_n(S)$ the permutation module of S_n afforded by the maximal chains of the rank-selected subposet $A^*_{n,k}(S)$. In this section we derive a recurrence for the action, and hence an explicit formula. We begin with an analogue of Theorem 3.3 for the chains.

PROPOSITION 5.1. *Let $\{P_n\}$ be any sequence of finite posets each carrying an action of the symmetric group S_n, such that for any $g \in S_n$, the fixed-point subposet P_n^g is isomorphic to the poset $P_{\text{fix}(g)}$, where $\text{fix}(g)$ is the number of fixed points of g as a permutation of S_n. Suppose that the number of maximal chains of P_n is a polynomial*

in $(n-1)$, *say* $\sum_{i \geqslant 0} a_i(n-1)^i$. *Then the permutation action of* S_n *on the maximal chains of* P_n *decomposes as a sum of* i*th tensor powers of the irreducible indexed by the partition* $(n-1,1), i \geqslant 0$, *with coefficient equal to* a_i. *In particular, the* S_n-*module structure of the maximal chains of* P_n *is completely determined by its dimension.*

Proof. Since S_n acts by permuting the chains, the trace of $g \in S_n$ on the chains of P_n is equal to the number of chains fixed by g. As in the proof of Theorem 3.3, the key point is that this in turn is the number of chains in the fixed-point poset P_n^g, and the latter coincides with $P_{\mathrm{fix}(g)}$. $\qquad\square$

Note that, as was the case with Theorem 3.3, Proposition 5.1 applies to all rank-selected subposets of $A_{n,k}^*$. Before we apply this, we state the following reformulation of an observation of Björner recorded in Theorem 2.5. Let $S = \{1 \leqslant s_1 < s_2 < \cdots < s_p \leqslant k\}$ be a subset of $[1,k]$. By Part (2) of Theorem 2.5, the number of words in $[\beta, \infty]$ depends only on $|\beta|$. This immediately gives the following recurrence for the dimensions of the modules $\alpha_n(S)$:

$$(11) \qquad \dim \alpha_n(S) = \dim \alpha_n(S \backslash \{s_p\}) \sum_{i=0}^{s_p - s_{p-1}} \binom{s_p}{i}(n-1)^i,$$

and $\dim \alpha_n(\{s_1\}) = n^{s_1} = \sum_{i=0}^{s_1} \binom{s_1}{i}(n-1)^i$.

THEOREM 5.2. *For any subset* $S \subseteq [1,k]$, *the* S_n-*module induced by the action of* S_n *on the maximal chains of the rank-selected subposet* $A_{n,k}^*(S)$ *is a* nonnegative *integer combination of tensor powers of the irreducible indexed by* $(n-1,1)$. *Hence the* S_n-*representation on the homology of the rank-selected subposet* $A_{n,k}^*(T), T \neq \varnothing$, *is an integer combination of positive tensor powers of the irreducible indexed by* $(n-1,1)$. *The highest tensor power that can occur is the* m*th, where* $m = \max(T)$.

Proof. Let $S = \{1 \leqslant s_1 < s_2 < \cdots < s_p \leqslant k\}$. From Proposition 5.1, it suffices to compute the dimension of the module of maximal chains in $A_{n,k}^*(S)$ as a polynomial in $(n-1)$. Hence (11) immediately gives the following recursive description for the modules $\alpha_n(S)$.

$$(12) \qquad \alpha_n(S) = \alpha_n(S \backslash \{s_p\}) \otimes \bigoplus_{i=0}^{s_p - s_{p-1}} \binom{s_p}{i} S_{(n-1,1)}^{\otimes i},$$

and

$$(13) \qquad \alpha_n(\{s_1\}) = \bigoplus_{i=0}^{s_1} \binom{s_1}{i} S_{(n-1,1)}^{\otimes i}.$$

By induction it is clear that $\alpha_n(S)$ is a nonnegative integer combination of $S_{(n-1,1)}^{\otimes j}, 0 \leqslant j \leqslant m = \max(S)$. It is also clear that the 0th tensor power, that is, the trivial module $S_{(n)}$, occurs exactly once in each $\alpha_n(S)$.

Note that when $S = \varnothing$, the homology is simply the trivial module. The claim about the decomposition of the homology into tensor powers of $S_{(n-1,1)}$ now follows from Stanley's theory of rank-selected homology representations [17]. We have

$$(14) \qquad \alpha_n(T) = \sum_{S \subseteq T} \beta_n(S) \quad \text{and thus} \quad \beta_n(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} \alpha_n(S),$$

where $\beta_n(S)$ is the representation of S_n on the homology of the rank-selected subposet $A_{n,k}^*(S)$ of $A_{n,k}^*$. When T is nonempty, it is clear from the previous paragraph that the occurrences of the 0th tensor power, which equals $S_{(n)}$, all cancel in (14); the trivial module occurs with coefficient $\sum_{S \subseteq T}(-1)^{|T|-|S|}$, which is zero. Hence only positive tensor powers will appear. $\qquad\square$

Thus Theorem 5.2 supports Conjecture 1.1. Note that it is easy to concoct signed integer combinations of tensor powers that are not true S_n-modules. For instance, the integer combination $S_{(n-1,1)}^{\otimes 2} - 2\,S_{(n-1,1)}$ decomposes into $S_{(n)} + S_{(n-2,2)} S_{(n-2,1,1)} - S_{(n-1,1)}$, while $S_{(n-1,1)}^{\otimes 2} - S_{(n-1,1)} = S_{(n)} + S_{(n-2,2)} + S_{(n-2,1,1)}$ is a true S_n-module. Also see Theorem 8.1 later in the paper.

It is worth pointing out the special case for the full poset $A_{n,k}^*$.

THEOREM 5.3. *The action of S_n on the maximal chains of $A_{n,k}^*$ decomposes into the direct sum of tensor powers*

$$S_{(n)} \oplus \bigoplus_{j=1}^{k} c(k+1,j) S_{(n-1,1)}^{\otimes k+1-j},$$

where $c(k+1,j)$ is the number of permutations in S_{k+1} with exactly j cycles in its disjoint cycle decomposition.

Proof. Specialising (11) to the case $S = [1,k]$ gives the recurrence $\dim \alpha_n([1,k]) = \dim \alpha_n([1,k-1])(1 + k(n-1))$, and clearly $\dim \alpha_n([1,1]) = n$. It follows that

$$\dim \alpha_n([1,k]) = \prod_{i=1}^{k}(1 + i(n-1)),$$

a formula due to Viennot [22, Lemma 4.1, Proposition 4.2].

Using the generating function (see [18]) $\sum_{j=1}^{m} c(m,j)t^j = t(t+1)(t+2)\ldots(t+(m-1))$, we find that

$$\dim \alpha_n([1,k]) = 1 + \sum_{j=1}^{k} c(k+1,j)(n-1)^{k+1-j}.$$

Invoking Proposition 5.1, the result follows, noting that the constant term in the above expression corresponds to the occurrence of the trivial representation. □

By expanding the expression for $\alpha_n(S)$ in (12), we have the following observation. Although $S_{(n-1,1)}$ is a quotient of two permutation modules, it is not clear how to deduce this corollary directly. Later in the paper we will examine these tensor powers more carefully; see Lemma 7.9.

COROLLARY 5.4. *The S_n-module $S_{(n)} \oplus \bigoplus_{j=1}^{k} c(k+1,j) S_{(n-1,1)}^{k+1-j}$ is in fact a permutation module. More generally, for any subset $S = \{1 \leqslant s_1 < \cdots < s_p \leqslant k\}$ of $[1,k]$, the S_n-module*

(15)
$$\bigotimes_{r=1}^{p} \left(\bigoplus_{i=0}^{s_r - s_{r-1}} \binom{s_r}{i} S_{(n-1,1)}^{\otimes i} \right), s_0 = 1,$$

is a permutation module.

Proof. The expression (15) gives the S_n-action on the chains of the rank-selected subposet $A_{n,k}^*(S)$ and is therefore a permutation module. □

By applying Proposition 5.1, we have the following two descriptions of the action on chains between two ranks.

PROPOSITION 5.5. *For $S = \{1 \leqslant s_1 < s_2 \leqslant k\}$, $\alpha_n(\{s_1 < s_2\})$ is given by*

(1)
$$\left(\bigoplus_{j=0}^{s_1} \binom{s_1}{j} S_{(n-1,1)}^{\otimes j} \right) \otimes \left(\bigoplus_{i=0}^{s_2 - s_1} \binom{s_2}{i} S_{(n-1,1)}^{\otimes i} \right)$$

(2) *and also by*

$$(S_{(n-1,1)} \oplus S_{(n)})^{\otimes s_1} \otimes \bigoplus_{j=0}^{s_2-s_1} \left[\binom{s_1+j-1}{j} S_{(n-1,1)}^{\otimes j} \bigotimes (S_{(n-1,1)} \oplus S_{(n)})^{\otimes s_2-s_1-j} \right].$$

Proof. From Proposition 5.1, it suffices to compute the dimension of the module of chains as a polynomial in $(n-1)$. From Equation (11), the dimension of $\alpha_n(\{s_1 < s_2\})$ is given by

(A) $$n^{s_1} \sum_{i=0}^{s_2-s_1} \binom{s_2}{i} (n-1)^i.$$

Recall Björner's generating function for the zeta function of subword order, Part (3) of Theorem 2.5, which we now use to count the number of chains from a fixed element to all elements above it of a fixed rank. This gives (by extracting the coefficient of t^{s_2} in the right-hand side of Part (3)):

(B) $$n^{s_1} \sum_{i,j \geqslant 0, i+j=s_2-s_1} \binom{s_1+j-1}{j} (n-1)^j n^i$$

$$= n^{s_1} \sum_{j=0}^{s_2-s_1} \binom{s_1+j-1}{j} (n-1)^j n^{s_2-s_1-j}.$$

Since as usual $(n-1)$ is the dimension of $S_{(n-1,1)}$ and n is the dimension of $S_{(n)} \oplus S_{(n-1,1)}$, invoking Proposition 5.1, we conclude that (A) and (B) correspond respectively to the S_n-module decompositions in Part (1) and Part (2). □

REMARK 5.6. By expanding in powers of $(n-1)$, the equivalence of the two expressions for the dimension of $\alpha_n(\{s_1 < s_2\})$ is equivalent to the following binomial coefficient identity:

$$\binom{a+r}{i} = \sum_{j=0}^{i} \binom{r-j}{i-j} \binom{a+j-1}{j} \text{ for } 0 \leqslant i \leqslant r,$$

or equivalently, putting $k = r - i$,

$$\binom{a+r}{a+k} = \sum_{j=0}^{r-k} \binom{r-j}{k} \binom{a+j-1}{a-1} \text{ for } 0 \leqslant k \leqslant r.$$

We can now show that Conjecture 1.1 is true for rank sets of size 2.

THEOREM 5.7. *Let $S = \{1 \leqslant s_1 < s_2 \leqslant k\}$ be a rank-set of size 2 in $A_{n,k}^*$. The homology representation of $A_{n,k}^*(S)$ is given by*

$$\sum_{v=1}^{s_1} S_{(n-1,1)}^{\otimes v} \left(c_v - \binom{s_1}{v} \right) \oplus \sum_{v=1+s_1}^{s_2} S_{(n-1,1)}^{\otimes v} c_v,$$

where c_v is the following positive integer:

$$c_v = \sum_{j=1}^{\min(v,s_2-s_1)} \binom{s_2-j}{v-j} \binom{s_1+j-1}{j}.$$

Moreover $c_v \geqslant \binom{s_1}{v}$ when $v \leqslant s_1$, and hence the homology is a nonnegative integer combination of positive tensor powers of $S_{(n-1,1)}$.

Proof. In order to establish the positivity, it is (curiously) easier to work with the second formulation of Proposition 5.5. By Theorem 3.3, it is enough to show that the dimension of the homology module is a polynomial in $(n-1)$ with nonnegative integer coefficients. We have

$$\beta_n(\{s_1 < s_2\}) = \alpha_n(\{s_1 < s_2\}) - \alpha_n(\{s_1\}) - \alpha_n(\{s_2\}) + \alpha_n(\varnothing)$$

which yields, (in terms of dimensions, from (B) in the proof of Proposition 5.5)

$$n^{s_1} \sum_{j=0}^{s_2-s_1} n^{s_2-s_1-j}(n-1)^j \binom{s_1+j-1}{j} - n^{s_2} - n^{s_1} + 1$$

$$= \sum_{j=1}^{s_2-s_1} n^{s_2-j}(n-1)^j \binom{s_1+j-1}{j} - (n^{s_1} - 1).$$

Expanding n^{s_1} and n^{s_2-j} in nonnegative powers of $(n-1)$ gives

$$\beta_n(\{s_1 < s_2\})$$

$$(16) \qquad = \sum_{j=1}^{s_2-s_1} \sum_{u=0}^{s_2-j} \binom{s_2-j}{u}(n-1)^{u+j}\binom{s_1+j-1}{j} - \sum_{j=1}^{s_1}\binom{s_1}{j}(n-1)^j$$

$$(17) \qquad = \sum_{v=1}^{s_2}(n-1)^v c_v - \sum_{j=1}^{s_1}\binom{s_1}{j}(n-1)^j,$$

where

$$c_v = \sum_{\substack{(u,j):u+j=v \\ 1\leqslant j\leqslant s_2-s_1,\, 0\leqslant u\leqslant s_2-j}} \binom{s_2-j}{v-j}\binom{s_1+j-1}{j}.$$

The latter sum runs over all j such $1 \leqslant j \leqslant s_2 - s_1$ and $0 \leqslant v - j \leqslant s_2 - j$, i.e. over all $j = 1, \ldots, \min(v, s_2 - s_1)$, as stated.

Now c_v is a sum of nonnegative integers for each $v = 1, \ldots, s_2$. When $v \leqslant s_1$, the $j = 1$ summand of c_v can be seen to be $\binom{s_2-1}{v-1}s_1$, and so

$$c_v - \binom{s_1}{v} \geqslant \binom{s_2-1}{v-1}s_1 - \binom{s_1}{v} = \frac{s_1!}{v!(s_2-v)!}\left(v\frac{(s_2-1)!}{(s_1-1)!} - \frac{(s_2-v)!}{(s_1-v)!}\right)$$

$$= \frac{s_1!(s_2-s_1)!}{v!(s_2-v)!}\left(v\binom{s_2-1}{s_2-s_1} - \binom{s_2-v}{s_2-s_1}\right)$$

and this is clearly nonnegative, since $\binom{s_2-1}{s_2-s_1} \geqslant \binom{s_2-v}{s_2-s_1}$ for $v \geqslant 1$. We have shown that the Betti number of $\beta_n(\{s_1 < s_2\})$ is a nonnegative integer combination of positive powers of $(n-1)$, as claimed. $\qquad\square$

REMARK 5.8. Let $s_1 = 1$, and consider the two ranks $\{1 < s_2\}$. The homology of the rank-selected subposet is then

$$\bigoplus_{v=2}^{s_2-1}\binom{s_2}{v-1}S_{(n-1,1)}^{\otimes v}\bigoplus(s_2-1)S_{(n-1,1)}^{\otimes s_2}.$$

6. Tensor powers of the reflection representation I

In this section we explore the tensor powers $S_{(n-1,1)}^{\otimes k}$. The paper [10] gives a combinatorial model for determining the multiplicity of an irreducible in the kth tensor power, and an explicit formula in the case when n is sufficiently larger than k. We give general formulas that apply to the case of arbitrary tensor powers.

We use symmetric functions to describe some of the results that follow. The homogeneous symmetric function h_n is the Frobenius characteristic, denoted ch, of the trivial representation of S_n. Also let $*$ denote the internal product on the ring of symmetric functions, so that the Frobenius characteristic of the Kronecker product of two S_n-modules is the internal product of the two characteristics. See [14, p. 115] and [19, Chapter 7, p. 476]. Recall that the natural representation of S_n is the permutation action on a set of n objects. The stabiliser of any one object is the Young subgroup $S_1 \times S_{n-1}$, and hence the natural representation is given by the induced module $1 \uparrow_{S_1 \times S_{n-1}}^{S_n}$, with Frobenius characteristic $h_1 h_{n-1}$. In particular we have the decomposition

$$1 \uparrow_{S_1 \times S_{n-1}}^{S_n} = S_{(n)} \oplus S_{(n-1,1)} = S_{(n,1)} \downarrow_{S_n}^{S_{n+1}} .$$

The following lemma is an easy exercise in permutation actions. We sketch a proof for completeness.

LEMMA 6.1. *Let $V_{j,n}$ denote the permutation module obtained from the S_n-action on the cosets of the Young subgroup $S_1^j \times S_{n-j}$. Then the kth tensor power of the natural representation $V_{1,n}$ of S_n decomposes into a sum of $S(k,j)$ copies of $V_{j,n}$, where $S(k,j)$ is the Stirling number of the second kind:*

$$(18) \quad V_{1,n}^{\otimes k} = \sum_{j=1}^{\min(n,k)} S(k,j) V_{j,n}, \quad \text{and thus} \quad (h_1 h_{n-1})^{*k} = \sum_{j=1}^{\min(n,k)} S(k,j) h_1^j h_{n-j}.$$

Proof. If the module $V_{1,n}$ is realised as $\mathbb{C}^n$ with basis $\{v_1 \ldots, v_n\}$, say, then $V_{1,n}^{\otimes k}$ is realised by the kth tensor power of $\mathbb{C}^n$, with n^k basis elements $v_{i_1} \otimes \cdots \otimes v_{i_k}$, $1 \leqslant i_1, \ldots, i_k \leqslant n$. The S_n-action now permutes these n^k basis elements. To determine the orbits, note that there is a surjection from this basis of tensors to the set partitions of a k-element set into nonempty blocks. Each such partition with j blocks indexes an orbit of the S_n-action, with stabiliser (conjugate to) $S_1^j \times S_{n-j}$. The blocks correspond to repetitions of a v_i in the tensor; thus a, b belong in the same block if $v_a = v_b$ in the tensor. The orbit is the transitive permutation representation with Frobenius characteristic $h_1^j h_{n-j}$.

The last statement is now immediate. □

EXAMPLE 6.2. We illustrate the above argument with an example. With $n = 5$ and $k = 7$, the tensor $v_5 \otimes v_2 \otimes v_2 \otimes v_4 \otimes v_2 \otimes v_4 \otimes v_5$ maps to the partition $17 - 235 - 46$ of a set of size $k = 7$ into $j = 3$ blocks, corresponding to the three distinct basis elements v_2, v_4, v_5 of $\mathbb{C}^n$. Its orbit under S_5 consists of all basis tensors $v_{i_1} \otimes \cdots \otimes v_{i_7}$ such that $v_{i_1} = v_{i_7}, v_{i_2} = v_{i_3} = v_{i_5}$, and $v_{i_4} = v_{i_6}$. Writing S_A for the permutations of the elements of A, for any subset A of positive integers, the stabiliser is $S_{\{5\}} \times S_{\{2\}} \times S_{\{4\}} \times S_{\{1,3\}}$, conjugate to the Young subgroup indexed by the integer partition $(2, 1, 1, 1)$ of 5.

REMARK 6.3. This lemma can also be proved by iterating a standard representation theory result, namely that for finite groups G and H with H a subgroup of G, and G-module W, H-module V, $W \otimes (V \uparrow_H^G) = (W \downarrow_H \otimes V) \uparrow_H^G$. In our case $G = S_n$ and H is the Young subgroup $S_1 \times S_{n-1}$.

THEOREM 6.4. *The top homology of $A_{n,k}^*$ has Frobenius characteristic*

$$\sum_{i=0}^{\min(n,k)} h_1^i h_{n-i} \left(\sum_{r=0}^{k-i} (-1)^r \binom{k}{r} S(k-r, i) \right).$$

Proof. Observe that the Frobenius characteristic of $(S_{(n-1,1)})^{\otimes k}$ is the k-fold internal product of $(h_1 h_{n-1} - h_n)$. Standard properties of the tensor product make $*$ a commutative and associative product in the ring of symmetric functions, so we have

$$(h_1 h_{n-1} - h_n)^{*k} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (h_1 h_{n-1})^{*j} * (h_n)^{*(k-j)}$$

$$= (-1)^k h_n + \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} (h_1 h_{n-1})^{*j}$$

$$= (-1)^k h_n + \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \sum_{i=1}^{\min(n,j)} S(j,i) h_1^i h_{n-i} \quad \text{from Lemma 6.1}$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{i=0}^{\min(n,j)} S(j,i) h_1^i h_{n-i}$$

$$= \sum_{i=0}^{\min(n,k)} h_1^i h_{n-i} \left(\sum_{j=i}^{k} (-1)^{k-j} \binom{k}{j} S(j,i) \right).$$

Note that $S(0,0) = 1$ and $S(j,0) = 0$ for $j \geqslant 1$. Putting $r = k - j$ in the last step gives the result. $\square$

Theorem 7.7 gives a different description of this module, from which it will be evident that the coefficients of $h_1^i h_{n-i}$ are positive for $i \geqslant 2$.

We can now determine the multiplicity of the trivial representation in the top homology of $A_{n,k}^*$:

COROLLARY 6.5. *Let $n \geqslant 2$. The following are equal:*

(1) *the multiplicity of the trivial representation in $S_{(n-1,1)}^{\otimes k}$;*
(2) *the multiplicity of the irreducible $S_{(n-1,1)}$ in $S_{(n-1,1)}^{\otimes k-1}$;*
(3) *the number*

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{i=0}^{\min(n,k)} S(k-r,i).$$

When $n \geqslant k$, this multiplicity equals the number of set partitions of $\{1, \ldots, k\}$ with no singleton blocks.

Proof. The first two multiplicities are equal by standard properties of the tensor product, since

$$\langle V \otimes W, S_{(n)} \rangle = \langle V, S_{(n)} \otimes W \rangle = \langle V, W \rangle.$$

The equivalence with the third formula follows from Theorem 6.4, since $\langle h_1^i h_{n-i}, h_n \rangle = 1$ for all i (alternatively, $1 \uparrow_{S_1^i \times S_{n-i}}^{S_n}$ is a transitive permutation module). Let $B_n^{\geqslant 2}$ denote the number of set partitions of $[n] = \{1, \ldots, n\}$ with no blocks of size 1, and let B_n denote the nth Bell number, that is, the total number of set partitions of $[n]$. Inclusion-exclusion shows that

$$(19) \qquad\qquad B_n^{\geqslant 2} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} B_{n-r},$$

since the number of partitions containing a fixed set of r singleton blocks is B_{k-r}. When $n \geqslant k$, the formula in Part (3) simplifies to

$$\sum_{r=0}^{k}(-1)^r \binom{k}{r} \sum_{i=0}^{k-r} S(k-r,i) = \sum_{r=0}^{k}(-1)^r \binom{k}{r} B_{k-r}.$$

That this number is $B_k^{\geqslant 2}$, the number of partitions of $[k]$ with no singleton blocks, now follows from Equation (19). (This is sequence A000296 in OEIS.) $\qquad\square$

Corollary 7.8 in the next section will give a different expression for the multiplicity of the trivial representation, for arbitrary n, k, as a sum of positive integers.

7. "Almost" an h-positive permutation module

We begin by recalling basic facts about permutation modules. V is a permutation module for S_n if there exists a basis for V that is permuted by the S_n-action. In particular the character values of a permutation module are all nonnegative. For example, the character $\chi_{(n-1,1)}$ of the reflection representation $S_{(n-1,1)}$ has values $\chi_{(n-1,1)}(g) = \text{fix}(g) - 1$ for each permutation $g \in S_n$, so it is negative for permutations without fixed points, and the same is true for odd tensor powers of $S_{(n-1,1)}$. Thus odd tensor powers of $S_{(n-1,1)}$ are not permutation modules. In particular, the homology of $A_{n,k}$ itself need not be a permutation module.

A special case of a permutation module occurs when its Frobenius characteristic is h-positive, that is, the coefficients in the basis of homogeneous symmetric functions are nonnegative. It is well known that the homogeneous symmetric function h_λ is the Frobenius characteristic of the transitive permutation representation whose orbit stabiliser is the Young subgroup of S_n indexed by λ. Hence the h-positivity of the Frobenius characteristic implies that it is a permutation module (whose point stabilisers are Young subgroups), but not conversely. A nice example is provided by the set partitions of $[4]$ into two blocks of size 2, viz. $12 - 34$, $13 - 24$, and $14 - 23$. The action of S_4 is a transitive permutation module whose point stabiliser is the wreath product $S_2[S_2]$, but its Frobenius characteristic is $h_4 + h_2^2 - h_1 h_3$, not h-positive.

The goal of this section is to prove the following theorem.

THEOREM 7.1. *Let $T \subseteq [1, k]$ be any nonempty subset of ranks in $A_{n,k}^*$. The following statements hold for the Frobenius characteristic $F_{n,k}(T)$ of the homology representation $\tilde{H}(A_{n,k}^*(T))$:*

   (1) *its expansion in the basis of homogeneous symmetric functions is an integer combination supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 1\}$.*
   (2) *$F_{n,k}(T) + (-1)^{|T|} s_{(n-1,1)}$ is supported on the set $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 2\}$.*

When $F_{n,k}(T) + (-1)^{|T|} s_{(n-1,1)}$ is in fact a nonnegative integer combination of $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 2\}$, we may view $F_{n,k}(T)$ as being almost a permutation module, hence the title of this section. First we prove a stronger result for the action on the chains.

THEOREM 7.2. *Let $S \subseteq [1, k]$. If $|S| \geqslant 1$, the S_n-module induced by the action of S_n on the space of chains $\alpha_n(S)$ has h-positive Frobenius characteristic supported on the set $T_1(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 1\}$. Furthermore, $h_1 h_{n-1}$ always appears with coefficient 1 in the h-expansion of $\alpha_n(S)$.*

Proof. Recall that $*$ denotes the inner tensor product. Note that the case of a single rank has already been established in Lemma 6.1: if $S = \{s_1\}$, then $\text{ch}\,\alpha_n(S) =$

$(h_1 h_{n-1})^{*s_1}$ since the dimension of the module is n^{s_1}, and the coefficient of $h_1 h_{n-1}$ in the h-expansion is the Stirling number $S(s_1, 1) = 1$.

We proceed by induction, using the decomposition (12) of Theorem 5.2. We have, with $|S| \geqslant 2$,

$$\alpha_n(S) = \alpha_n(S \backslash \{s_p\}) \otimes \sum_{i=0}^{s_p - s_{p-1}} \binom{s_p}{i} S_{(n-1,1)}^{\otimes i}$$

$$= \alpha_n(S \backslash \{s_p\}) \otimes \left(\sum_{i=1}^{s_p - s_{p-1}} \binom{s_p}{i} S_{(n-1,1)}^{\otimes i} \oplus S_{(n)} \right).$$

Translating Remark 6.3 into Frobenius characteristics gives the well-known symmetric function formula $(h_1 h_{n-1}) * f = h_1 \frac{\partial}{\partial p_1} f$ for any symmetric function f of homogeneous degree n [19, Exercise 7.81, p. 477], [14, Example 3 (c), p. 75]. It is easy to check that for $a \geqslant 1, a + b = n$,

$$(20) \qquad (h_1^a h_b) * s_{(n-1,1)} = \begin{cases} (a-1)h_1^a h_b + h_1^{a+1} h_{b-1}, & a \geqslant 2; \\ h_1^{a+1} h_{b-1}, & a = 1. \end{cases}$$

Note the factor h_1^2 when $a \neq 0$. Let $T_2(n) = \{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 2\}$. Iterating (20) gives the fact that when $\mu \in T_1(n-1)$, $(h_1 h_\mu) * s_{(n-1,1)}^{*k}$ is a nonnegative integer combination of terms in $T_2(n)$.

Assume $S = \{1 \leqslant s_1 < \cdots < s_p \leqslant k\}$, and $|S| \geqslant 2$. In terms of symmetric functions, the decomposition (12) of Theorem 5.2 becomes

$$(21) \qquad \operatorname{ch} \alpha_n(S) = \operatorname{ch} \alpha_n(S \backslash \{s_p\}) * \left(h_n + \sum_{i=1}^{s_p - s_{p-1}} \binom{s_p}{i} s_{(n-1,1)}^{*i} \right)$$

$$= \operatorname{ch} \alpha_n(S \backslash \{s_p\}) + \operatorname{ch} \alpha_n(S \backslash \{s_p\}) * \left(\sum_{i=1}^{s_p - s_{p-1}} \binom{s_p}{i} s_{(n-1,1)}^{*i} \right).$$

Suppose now that the first term above, $\operatorname{ch} \alpha_n(S \backslash \{s_p\})$, is a nonnegative integer combination of terms in $T_1(n)$, in which $h_1 h_{n-1}$ appears with coefficient 1. By (20), the h-expansion of the second term contains only terms in $T_2(n)$; the crucial point here is that, since $p \geqslant 2$, the h-expansion of $\operatorname{ch} \alpha_n(S \backslash \{s_p\})$ does not contain the function h_n. It follows that $\operatorname{ch} \alpha_n(S)$ must be a nonnegative integer combination of terms in $T_1(n)$, and the coefficient of $h_1 h_{n-1}$ is inherited from $\operatorname{ch} \alpha_n(S \backslash \{s_p\})$. It is therefore equal to 1. This completes the induction. $\qquad \square$

Proof of Theorem 7.1. Using Stanley's equation for rank-selected homology, Equation (14) in Theorem 5.2, we have

$$F_{n,k}(T) = \sum_{S \subseteq T} (-1)^{|T| - |S|} \operatorname{ch} \alpha_n(S).$$

From Theorem 7.2, this has an expansion in the h-basis in which h_n appears only in $\alpha_n(\varnothing)$ with coefficient 1, and $h_1 h_{n-1}$ appears in $\alpha_n(S)$ with coefficient 1 for all nonempty S. Hence the coefficient of h_n in the right-hand side above is $(-1)^{|T|}$, while the coefficient of $h_1 h_{n-1}$ is

$$\sum_{S \subseteq T, S \neq \varnothing} (-1)^{|T| - |S|} = (-1)^{|T|} \left[\sum_{i=0}^{|T|} \binom{|T|}{i} (-1) - 1 \right] = (-1)^{|T| - 1}.$$

But $(-1)^{|T|} s_{(n-1,1)} = (-1)^{|T|} h_1 h_{n-1} - (-1)^{|T|} h_n$, and the conclusion follows. $\qquad \square$

The preceding theorem motivates Conjecture 1.2 in the Introduction. We will show that Conjecture 1.2 is true in the following cases of rank-selection:

THEOREM 7.3. *For any nonempty rank set $T \subseteq [1, k]$, consider the module $V_T = \tilde{H}_{k-2}(A^*_{n,k}(T)) + (-1)^{|T|}S_{(n-1,1)}$. In each of the following cases, V_T is a nonnegative combination of transitive permutation modules with orbit stabilisers of the form $S^d_1 \times S_{n-d}, d \geqslant 2$. Equivalently, the symmetric function $F_{n,k}(T) + (-1)^{|T|}s_{(n-1,1)}$ is supported on the set $T_2(n) = \{h_\lambda : \lambda = (n - r, 1^r), r \geqslant 2\}$ with nonnegative integer coefficients in each of the following cases:*

(1) $T = [r, k], k \geqslant r \geqslant 1$.
(2) $T = [1, k]\backslash\{r\}, k \geqslant r \geqslant 1$.
(3) $T = \{1 \leqslant s_1 < s_2 \leqslant k\}$.

The proof relies on the homology computations of the preceding sections, but we also need to develop additional tools. In particular Theorem 7.7 will be crucial to the proof. We begin by deriving a different expression for the Whitney homology modules of $A^*_{n,k}$, thereby obtaining a new expression for the top homology module as well.

PROPOSITION 7.4. *Let α be a nonempty word in $A^*_{n,k}$. Then*

(1) *If α is not a normal word, the (order complex of the) interval $(\hat{0}, \alpha)$ is contractible and hence its homology vanishes in all degrees;*
(2) *If α is a normal word, the (order complex of the) interval $(\hat{0}, \alpha)$ is homotopy equivalent to a single sphere in the top dimension, and the stabiliser subgroup of α acts trivially on the homology.*

Proof. The topological conclusions in both parts are immediate from the formula for the Möbius number in Theorem 2.2 and Björner's dual CL-shellability result of Theorem 2.6.

If α is normal, then it consists of some $m \leqslant n$ distinct letters $\{x_1, \ldots, x_m\}$, and consecutive letters are distinct. The stabiliser is the subgroup which fixes each x_i pointwise, and permutes the remaining $n - m$ letters arbitrarily. It is thus a product of m copies of the trivial group S_1 and the group S_{n-m}. Clearly this subgroup fixes every element in the interval $(\hat{0}, \alpha)$ pointwise, and hence the action on the unique nonvanishing homology is trivial. □

Let $S^*(j, d)$ denote the number of set partitions of $[j]$ into d blocks, with the property that no block contains consecutive integers (a reduced Stirling number). The surjection in the following lemma will be needed in what follows.

LEMMA 7.5. *There is a surjection ψ from the set of words of length j in an alphabet of size n to the set partitions into d blocks of $[j]$, where d is the number of distinct letters in α. This surjection maps normal words onto set partitions with the property that no two consecutive integers are in the same block. In particular, the number $n(n - 1)^{j-1}$ of normal words of length $j \geqslant 2$ on an alphabet of size n is also equal to*

$$\sum_{d=1}^{\min(n,j)} \frac{n!}{(n-d)!} S^*(j, d).$$

Proof. The idea of the proof is similar to that of Lemma 6.1. A word α of length j with d distinct letters maps to a set partition $\psi(\alpha)$ of $[j]$ with d blocks, where positions corresponding to integers in the same block have equal letters in α. For instance, the word $\alpha = abbcbca$, of length 7 with 3 distinct letters. Then the set partition of $[7]$ associated to α is $\psi(\alpha) = 17 - 235 - 46$.

Now suppose α is normal. Then no two consecutive positions have equal letters, which is precisely the condition that no block of the set partition $\psi(\alpha)$ contains consecutive integers.

The last statement is verified by observing that a normal word in the pre-image of every set partition of $[j]$ with d blocks contains d distinct letters chosen out of n, which can be permuted amongst themselves in d ways. $\square$

Recall that the ordinary Stirling numbers of the second kind satisfy the recurrence $S(n+1, d) = S(n, d-1) + dS(n, d)$ with initial conditions $S(0, 0) = 1$ and $S(n, 0) = 0 = S(0, d)$ if $n, d > 0$. It is easy to verify similarly that the reduced Stirling numbers $S^*(j, d)$ satisfy the recurrence $S^*(n+1, d) = S^*(n, d-1) + (d-1)S^*(n, d)$, by examining the possibilities for inserting $(n+1)$ into a partition of $[n]$ into d blocks. A comparison of the recurrences immediately shows that in fact

$$S^*(n+1, d) = S(n, d-1) \quad \text{for all } n \geqslant 0, d \geqslant 1.$$

See [15] for generalisations of these numbers. Recall that in Theorem 3.5, the jth Whitney homology of $A_{n,k}^*$ $j \geqslant 2$, was determined as a sum of two consecutive tensor powers of $S_{(n-1,1)}$. From Lemma 7.5 and Proposition 7.4 we now have the following surprising result.

PROPOSITION 7.6. *Each Whitney homology module of subword order, and hence the sum of two consecutive tensor powers of the reflection representation, has h-positive Frobenius characteristic, and in particular it is a permutation module. We have* $\operatorname{ch} WH_0 = h_n, \operatorname{ch} WH_1 = h_1 h_{n-1}$, *and for* $k \geqslant j \geqslant 2$, *the jth Whitney homology of* $A_{n,k}^*$ *has Frobenius characteristic*

$$
\begin{aligned}
(22) \qquad (h_1 h_{n-1}) * s_{(n-1,1)}^{*(j-1)} &= \sum_{d=2}^{j} S(j-1, d-1)\, h_1^d h_{n-d} \\
&= \sum_{d=2}^{j} S^*(j, d)\, h_1^d h_{n-d} = h_1(h_1 h_{n-2})^{*j-1},
\end{aligned}
$$

a permutation module with orbits whose stabilisers are Young subgroups indexed by partitions of the form $(n-d, 1^d), d \geqslant 0$.

Proof. From Equation (6) and Theorem 3.5, for $k \geqslant j \geqslant 2$, we have

$$WH_j(A_{n,k}^*) = S_{(n-1,1)}^{\otimes j} \oplus S_{(n-1,1)}^{\otimes j-1} = (S_{(n)} \oplus S_{(n-1,1)}) \otimes S_{(n-1,1)}^{\otimes j-1}.$$

Now by definition we also have

$$WH_j(A_{n,k}^*) = \sum_{x \in A_{n,k}^*, |x|=j} \tilde{H}(\hat{0}, x).$$

Proposition 7.4 says that the sum runs over only normal words x, and each homology module is trivial for the stabiliser of x. Collecting the summands into orbits and using the surjection of Lemma 7.5 gives Equation (22). The last expression is obtained by shifting the index in the sum:

$$\sum_{d=2}^{j} S(j-1, d-1)\, h_1^d h_{n-d} = h_1 \sum_{d'=1}^{j-1} S(j-1, d') h_1^{d'} h_{(n-1)-d'},$$

and this equals $h_1(h_1 h_{n-2})^{*j-1}$ by Lemma 6.1. $\square$

Recall [14] that the homogeneous symmetric functions h_λ form a basis for the ring of symmetric functions.

THEOREM 7.7. *Fix $k \geqslant 1$. The kth tensor power of the reflection representation $S_{(n-1,1)}^{\otimes k}$, i.e. the homology module $\tilde{H}_{k-1}(A_{n,k}^*)$, has the following property: $S_{(n-1,1)}^{\otimes k} \oplus (-1)^k S_{(n-1,1)}$ is a permutation module $U_{n,k}$ whose Frobenius characteristic is h-positive, and is supported on the set $\{h_\lambda : \lambda = (n-r,1^r), r \geqslant 2\}$. If $k = 1$, then $U_{n,1} = 0$.*

More precisely, the k-fold internal product $s_{(n-1,1)}^{*k}$ has the following expansion in the basis of homogeneous symmetric functions h_λ:

(23)
$$\sum_{d=0}^{n} g_n(k,d) h_1^d h_{n-d},$$

where $g_n(k,0) = (-1)^k, g_n(k,1) = (-1)^{k-1}$, and

$$g_n(k,d) = \sum_{i=d}^{k} (-1)^{k-i} S(i-1, d-1), \text{ for } 2 \leqslant d \leqslant n.$$

Hence

$$s_{(n-1,1)}^{*k} = (-1)^{k-1} s_{(n-1,1)} + \mathrm{ch}(U_{n,k}),$$

where $\mathrm{ch}(U_{n,k}) = \sum_{d=2}^{n} g_n(k,d) h_1^d h_{n-d}$.

The integers $g_n(k,d)$ are independent of n for $k \leqslant n$, nonnegative for $2 \leqslant d \leqslant k$, and $g_n(k,d) = 0$ if $d > k$. Also:

(1) $g_n(k,2) = \frac{1+(-1)^k}{2}$.
(2) $g_n(k,k-1) = \binom{k-1}{2} - 1, k \leqslant n$.
(3) $g_n(k,k) = 1, k < n$.

In particular the coefficient of h_1^n in the expansion (23) of $\mathrm{ch}\, S_{(n-1,1)}^{\otimes k}$ is

$$\begin{cases} g_n(k,n) + g_n(k,n-1) & \text{if } k > n, \\ \binom{n-1}{2} & \text{if } k = n, \\ 1 & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $k = 1$, the terms for $d \geqslant 2$ in the summation in (23) vanish and thus the right-hand side equals the characteristic of the top homology.

The first statement, about the homology module $\tilde{H}_{k-1}(A_{n,k}^*)$, follows from Proposition 7.6 and Theorem 3.1. Fix m and d such that $m \geqslant d \geqslant 2$. Let $\bar{g}(m,d)$ be the alternating sum of Stirling numbers $\bar{g}(m,d) = \sum_{i=d}^{m} (-1)^{m-i} S(i-1, d-1)$. Note that $\bar{g}(m,d)$ equals

$$[S(m-1,d-1) - S(m-2,d-1)] + [S(m-3,d-1) - S(m-4,d-1)] + \cdots$$

$$\cdots + \begin{cases} [S(d+1,d-1) - S(d,d-1)] + S(d-1,d-1), & m-d \text{ even,} \\ [S(d,d-1)) - S(d-1,d-1)], & m-d \text{ odd.} \end{cases}$$

Since $S(n,d)$ is an increasing function of $n \geqslant d$ for fixed d, the coefficient $\bar{g}(m,d)$ is always nonnegative. It is also clear that $\bar{g}(m,d) = S(m-1,d-1) - \bar{g}(m-1,d)$ for all $m \geqslant d \geqslant 2$.

The remaining parts follow from the facts that $S(k,k-1) = \binom{k}{2}, S(k,k) = 1$, and the observation that for $k \geqslant n$, the coefficient of h_1^n is $g_k(k,n) + g_k(k,n-1)$. This equals $S(n-1,n-2) - 1$ when $k = n$. $\qquad\square$

COROLLARY 7.8. *Let $k \geqslant 2$.*

(1) *For* $\min(n, k) \geqslant d \geqslant 2$, *the coefficient of* $h_1^d h_{n-d}$ *in* $s_{(n-1,1)}^{*k} = \operatorname{ch} S_{(n-1,1)}^{\otimes k}$ *is the nonnegative integer* $g_n(k, d)$ *given by the two equal expressions:*

$$(24) \qquad \sum_{j=d}^{k} (-1)^{k-j} S(j-1, d-1) = \sum_{r=0}^{k-d} (-1)^r \binom{k}{k-r} S(k-r, d).$$

In particular, when $n \geqslant k$, *this multiplicity is independent of* n.

(2) *The positive integer* $\beta_n(k) = \sum_{d=2}^{\min(n,k)} g_n(k, d)$ *is the multiplicity of the trivial representation in* $S_{(n-1,1)}^{\otimes k}$. *When* $n \geqslant k$, *it equals the number of set partitions* $B_k^{\geqslant 2}$ *of the set* $\{1, \ldots, k\}$ *with no singleton blocks. We have* $\beta_n(n+1) = B_{n+1}^{\geqslant 2} - 1$ *and* $\beta_n(n+2) = B_{n+2}^{\geqslant 2} - \binom{n+1}{2}$.

Proof. This follows from Theorem 6.4 and Corollary 6.5.

We have $\beta_n(n) = \sum_{d=2}^{n} g(n, d) = B_n^{\geqslant 2} = \beta_n(k)$ for $n \geqslant k$, and from (23),

$$\beta_n(n+1) = \sum_{d=2}^{n} g_n(n+1, d) = \sum_{d=2}^{n+1} g_{n+1}(n+1, d) - g_{n+1}(n+1, n+1)$$

$$= B_{n+1}^{\geqslant 2} - 1,$$

$$\beta_n(n+2) = \sum_{d=2}^{n+2} g_{n+2}(n+2, d) - g_{n+2}(n+2, n+2) - g_{n+2}(n+2, n+1)$$

$$= B_{n+2}^{\geqslant 2} - 1 - \left[\binom{n+1}{2} - 1 \right] = B_{n+2}^{\geqslant 2} - \binom{n+1}{2}. \qquad \square$$

We need one final observation in order to prove Theorem 7.3.

Lemma 7.9. *Suppose* V *is an* S_n-*module which can be written as an integer combination* $V = \oplus_{k=1}^{m} c_k S_{(n-1,1)}^{\otimes k}$ *of positive tensor powers of* $S_{(n-1,1)}$. *Then*

(1) *The character value of* V *on fixed-point-free permutations is* $\sum_{k=1}^{m} (-1)^k c_k$.
(2) *If* $\sum_{k=1}^{m} (-1)^{k-1} c_k = 0$, *then the Frobenius characteristic of* V *is supported on the set* $\{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 2\}$.
(3) *If* $\sum_{k=1}^{m} (-1)^{k-1} c_k = 0$ *and* $c_k \geqslant 0$ *for all* $k \geqslant 2$, *it is* h-*positive and hence* V *is a permutation module.*

Proof. The first part follows because the value of the character of the reflection representation $S_{(n-1,1)}$ on permutations without fixed points is (-1).

The second and third parts are immediate from Theorem 7.7, since we have

$$V = \left(\sum_{k=1}^{m} (-1)^{k-1} c_k \right) S_{(n-1,1)} \oplus \sum_{k=2}^{m} c_k U_{n,k} = \sum_{k=2}^{m} c_k U_{n,k},$$

and $U_{n,k}$ is h-positive with support $\{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 2\}$. Note that $U_{n,1} = 0$. $\square$

In particular from Theorem 5.3, this gives a direct proof that the action of S_n on the chains in $A_{n,k}^*$ is also a nonnegative linear combination of $\{h_\lambda : \lambda = (n-r, 1^r), r \geqslant 2\}$.

Proof of Theorem 7.3. Note that in all three cases, the module $V_T = \tilde{H}_{k-2}(A_{n,k}^*(T)) + (-1)^{|T|} S_{(n-1,1)}$ has already been shown to be an integer combination of kth tensor powers of $S_{(n-1,1)}$, with nonnegative coefficients when $k \geqslant 2$, in Theorems 3.7, 4.2 and 5.7. Hence, by Lemma 7.9, it remains only to verify that the alternating sum of coefficients of the tensor powers vanishes for V_T in each case.

Consider the case $T = [r, k]$. From Theorem 3.7, we must show that $(-1)^{k-r+1}$ added to the signed sum of the $(-1)^{i-1}b_i$, for the coefficients $b_i = \binom{k}{i}\binom{i-1}{k-r}$, is zero, i.e.

$$(25) \qquad \sum_{i=1+k-r}^{k} b_i(-1)^{i-1} = (-1)^{k-r}.$$

It is easiest to use the combinatorial identity of Corollary 3.8. Consider the two polynomials of degree $k \geqslant 2$ in x defined by

$$F(x) = \sum_{i=0}^{k-r}(-1)^i\binom{k}{r+i}(x+1)^{r+i}x^{k-(r+i)} + (-1)^{k+1-r},$$

$$G(x) = \sum_{i=1+k-r}^{k}\binom{k}{i}\binom{i-1}{k-r}x^i.$$

Corollary 3.8 says $F(x)$ and $G(x)$ agree for all $x = n - 1 \geqslant 1$, and hence $F(x) = G(x)$ identically.

In particular $F(-1) = G(-1)$. But $F(-1) = (-1)^{k+r-1}$ and clearly $(-1)G(-1)$ is precisely the sum in the left-hand side of (25). The claim follows.

For the second case, T is the rank-set $[1, k] \backslash \{r\}$, and from Theorem 4.2 the alternating sum of coefficients in V_T is clearly

$$(26) \qquad (-1)^{k-1} + \left[\binom{k}{r} - 1\right](-1)^{k-1} + \binom{k}{r}(-1)^{k-2} = 0.$$

For the final case, the rank set is $T = \{1 \leqslant s_1 < s_2 \leqslant k\}$. From the homology formula in Theorem 5.7, we need to show that the following sum, the alternating sum of coefficients in V_T, vanishes:

$$1 + \sum_{v=1}^{s_1}(-1)^{v-1}\left[c_v - \binom{s_1}{v}\right] + \sum_{v=1+s_1}^{s_2}(-1)^{v-1}c_v,$$

where

$$c_v = \sum_{j=1}^{\min(v, s_2-s_1)}\binom{s_2 - j}{v - j}\binom{s_1 + j - 1}{j}.$$

But $1 + \sum_{v=1}^{s_1}(-1)^{v-1}(-\binom{s_1}{v}) = 0$, so this reduces to showing that $\sum_{v=1}^{s_2}(-1)^v c_v = 0$. Split the summation over v at $s_2 - s_1$. This gives that $\sum_{v=1}^{s_2}(-1)^v c_v$ equals

$$\underbrace{\sum_{v=1}^{s_2-s_1}(-1)^v\sum_{j=1}^{v}\binom{s_2 - j}{v - j}\binom{s_1 + j - 1}{j}}_{(A)} + \underbrace{\sum_{v>s_2-s_1}^{s_2}(-1)^v\sum_{j=1}^{s_2-s_1}\binom{s_2 - j}{v - j}\binom{s_1 + j - 1}{j}}_{(B)}.$$

Switching the order of summation, (A) is equal to

$$\sum_{j=1}^{s_2-s_1}\binom{s_1 + j - 1}{j}\sum_{v=j}^{s_2-s_1}\binom{s_2 - j}{v - j}(-1)^v,$$

while (B) is

$$\sum_{j=1}^{s_2-s_1}\binom{s_1 + j - 1}{j}\sum_{v>s_2-s_1}^{s_2}(-1)^v\binom{s_2 - j}{v - j}(-1)^v.$$

Hence $\sum_{v=1}^{s_2}(-1)^v c_v$ equals

$$\sum_{j=1}^{s_2-s_1}\binom{s_1+j-1}{j}\sum_{v=j}^{s_2}\binom{s_2-j}{v-j}(-1)^v = \sum_{j=1}^{s_2-s_1}\binom{s_1+j-1}{j}\sum_{w=0}^{s_2-j}\binom{s_2-j}{w}(-1)^{w-s_2},$$

where we have put $w = s_2 - v$. But $1 \leqslant j \leqslant s_2 - s_1 < s_2$, so the inner sum vanishes. $\square$

Note that the left-hand side of Equation (25) is the character value on fixed-point-free permutations for the homology module $\tilde{H}(T)$, $T = [r, k]$. Hence this shows that the homology module itself cannot be a permutation module when $k - r$ is odd, since the right-hand side then gives a value of (-1) for the character.

Similarly, from Theorem 4.2, the character value on fixed-point-free permutations for $\tilde{H}(T)$, for $T = [1, k] \backslash \{r\}$, is $\left[\binom{k}{r} - 1\right](-1)^k + \binom{k}{r}(-1)^{k-1}$, and this equals $(-1)^{k-1}$. Once again we can conclude that this homology module is also not a permutation module when k is even.

Corollary 7.10. *The dual Whitney homology modules $WH_{k+1-i}^*(A_{n,k}^*), 1 < i \leqslant k$, are permutation modules whose Frobenius characteristic is a nonnegative integer combination of the set $T_2 = \{h_\lambda : \lambda = (n - r, 1^r), r \geqslant 2\}$.*

Proof. From Theorem 3.1 and (5), we have $WH_{k+1-i}^* = \tilde{H}([i, k]) \oplus \tilde{H}([i + 1, k])$. This in turn can be rewritten as

$$WH_{k+1-i}^* = \left(\tilde{H}([i, k]) + (-1)^{k-i+1}S_{(n-1,1)}\right) \oplus \left(\tilde{H}([i + 1, k]) + (-1)^{k-i}S_{(n-1,1)}\right);$$

by Part (2) of Theorem 7.3, each of the summands in parentheses is h-positive with Frobenius characteristic supported by the set T_2. $\square$

8. Tensor powers of the reflection representation II

In light of the preceding results, in this section we return to consider the tensor powers $S_{(n-1,1)}^{\otimes k}$ more closely. The paper [10] gives a combinatorial model for determining the multiplicity of an irreducible in the kth tensor power, and an explicit formula in the case when n is sufficiently larger than k. We give general formulas that apply to the case of arbitrary tensor powers.

Burnside proved that given a faithful representation V of a finite group G, every G-irreducible occurs in some tensor power of V. A simple and beautiful proof of a generalisation of this was given by Brauer in [8]. See also [11, Theorem 4.3]. In the present context, it states that since $S_{(n-1,1)}$ is a faithful representation whose character takes on n distinct values (viz. $-1, 0, 1, \ldots, n - 1$, but not $n - 2$), every irreducible S_λ occurs in at least one of the n tensor powers $S_{(n-1,1)}^{\otimes j}, 0 \leqslant j \leqslant n - 1$.

In view of these results, the next fact is interesting. We were unable to find it in the literature. Our proof mimics Brauer's elegant argument.

Theorem 8.1. *Let G be any finite group and X any character of G. Suppose X takes on k distinct nonzero values $b_1, \ldots, b_k$. Then the first k tensor powers of X are linearly independent functions on G, and form a basis for the subspace of class functions spanned by all the positive tensor powers. If $X^{k+1} = \sum_{i=1}^k c_i X^i$, then the polynomial $P(t) = t^{k+1} - \sum_{i=1}^k c_i t^i$ has the factorisation $t \prod_{i=1}^k (t - b_i)$.*

Let $X^0 = 1_G$ denote the trivial character of G. Then $X^k = \sum_{i=1}^k c_i X^{i-1}$ if and only if the character X never takes the value zero.

Proof. Let U be the vector space spanned by the characters X^j of the positive tensor powers of X. Suppose X takes the distinct values $\{b_i \neq 0 : 1 \leqslant i \leqslant k\}$. For each $i = 1, \ldots, k$, choose an arbitrary element in the preimage of b_i, that is, $a_i \in X^{-1}(b_i)$.

Let $A = \{a_i : 1 \leqslant i \leqslant k\}$. We may thus view U as a subspace of the space of functions defined on the set A of size k; this space has dimension exactly k, and hence $\dim(U) \leqslant k$. We will show that the characters $X^i, 1 \leqslant i \leqslant k$, are linearly independent.

We now claim that the k functions

$$\{X^j \downarrow_A, 1 \leqslant j \leqslant k\}$$

are linearly independent. Suppose c_j are scalars such that $\sum_{j=1}^{k} c_j X^j$ is the zero function. This implies $\sum_{j=1}^{k} c_j X^j(a_j) = 0$. But $X^j(a_i) = b_i^j$, so the coefficient matrix $(X^j(a_i))$ is a k by k Vandermonde with determinant $(b_1 \ldots b_k) \prod_{1 \leqslant i < j \leqslant k}(b_j - b_i)$, which is nonzero by hypothesis. Hence $c_j = 0$ for $1 \leqslant j \leqslant k$. This establishes the first statement.

Now let $X^{k+1} = \sum_{i=1}^{k} c_i X^i$ for some scalars c_i. If X never takes on the value zero we can clearly simplify the dependence relation to $X^k = \sum_{i=1}^{k} c_i X^{i-1}$. If however 0 is a value of X, the set $\{1_G, X^i : 1 \leqslant i \leqslant k\}$ must be linearly independent, since the trivial character equals 1 everywhere. This finishes the proof. $\qquad \square$

REMARK 8.2. For $k \geqslant 2$, the representation $S_{(n-1,1)}^{\otimes(k-1)}$ is contained in $S_{(n-1,1)}^{\otimes k}$. For this it suffices to note that

(1) this is true for $k = 2$, since $S_{(n-1,1)}^{\otimes 2} - S_{(n-1,1)} = S_{(n-2,2)} \oplus S_{(n-1,1^2)} \oplus S_{(n)}$, and thus

(2) $S_{(n-1,1)}^{\otimes k} - S_{(n-1,1)}^{\otimes(k-1)} = S_{(n-1,1)}^{\otimes(k-2)} \otimes (S_{(n-1,1)}^{\otimes 2} - S_{(n-1,1)})$ is a true module.

EXAMPLE 8.3. Write X_n^k for $S_{(n-1,1)}^{\otimes k}$. Maple computations with Stembridge's SF package show that

(1) $X_3^3 = X_3^2 + 2X_3$.
(2) $X_4^4 = 3X_4^3 + X_4^2 - 3X_4$.
(3) $X_5^5 = 6X_5^4 - 7X_5^3 - 6X_5^2 + 8X_5$.
(4) $X_6^6 = 10X_6^5 - 30X_6^4 + 20X_6^3 + 31X_6^2 - 30X_6$
(5) $X_7^7 = 15X_7^6 - 79X_7^5 + 165X_7^4 - 64X_7^3 - 180X_7^2 + 144X_7$
(6) $X_8^8 = 21X_8^7 - 168X_8^6 + 630X_8^5 - 1029X_8^4 + 189X_8^3 + 1198X_8^2 - 840X_8$.

We conclude this section by pointing out a representation-theoretic consequence, and some enumerative implications, of Theorem 7.7 and in particular of the expansion (23). Fix $n \geqslant 3$ and consider the n by $n - 1$ matrix D_n whose kth column consists of the coefficients $g_n(n - k, n - d), d = 1, \ldots, n - 1$. Thus the kth column contains the coefficients in the expansion of $S_{(n-1,1)}^{\otimes n-k}$ in the h-basis: we have $\operatorname{ch} S_{(n-1,1)}^{\otimes k} = \sum_{d=1}^{n} g_n(k, n-d) h_1^{n-d} h_d, 1 \leqslant k \leqslant n-1$. From Theorem 7.7 it is easy to see that the matrix D_n has rank $(n-1)$; the last two rows, consisting of alternating ± 1s, differ by a factor of (-1), and the matrix is lower triangular with 1's on the diagonal, hence it has rank $(n-1)$. Similarly the $(n+1)$ by $(n-1)$ matrix obtained by appending to D_n a first column consisting of the h-expansion of the nth tensor power of $S_{(n-1,1)}^{\otimes n}$ also has rank $(n-1)$. We therefore have a second proof of Theorem 8.1, for the specific case of the modules $S_{(n-1,1)}$. In this special case we can now be more precise about the linear combination of tensor powers:

THEOREM 8.4. *The first $n - 1$ tensor powers of $S_{(n-1,1)}$ are an integral basis for the vector space spanned by the positive tensor powers. The nth tensor power of $S_{(n-1,1)}$ is an integer linear combination of the first $(n-1)$ tensor powers:*

$$S_{(n-1,1)}^{\otimes n} = \bigoplus_{k=1}^{n-1} a_k(n) S_{(n-1,1)}^{\otimes k},$$

with $a_{n-1}(n) = \binom{n-1}{2}$. The coefficients $a_k(n)$ are determined by the polynomial $P(t) = t^n - \sum_{k=1}^{n-1} a_k(n)t^k$, defined by

$$(27) \qquad P(t) = \frac{t+1}{t-(n-2)} \sum_{j=1}^{n} c(n,j)t^j(-1)^{n-j}$$

where $c(n,j)$ is the number of permutations in S_n with exactly j disjoint cycles.

Proof. We invoke Theorem 8.1. The linear combination of tensor powers in the statement translates into a polynomial equation for the character values, whose zeros are the n distinct values $-1, 0, 1, \ldots, n-3, n-1$, taken by the character of $S_{(n-1,1)}$. Hence we have

$$P_n(t) = t^n - \sum_{k=1}^{n-1} a_k(n)t^k = (t+1)t \prod_{i=1, i \neq n-2}^{n-1} (t-i) = \frac{t+1}{(t-(n-2))} \prod_{i=0}^{n-1} (t-i).$$

But $\prod_{j=0}^{n-1}(t-j)$ is the generating function for the signless Stirling numbers of the first kind [18], so the result follows. $\qquad\square$

The preceding result gives a recurrence for the coefficients $a_k(n)$; we have

$$a_{n-1}(n) = \binom{n-1}{2};$$
$$(n-2)a_j(n) - a_{j-1}(n) = (-1)^{n-j}[c(n,j) - c(n,j-1)], 2 \leqslant j \leqslant n-1;$$
$$(n-2)a_1(n) = c(n,1)(-1)^{n-1}$$
$$\implies a_1(n) = \frac{(n-1)!}{n-2}(-1)^{n-1} = (-1)^{n-1}[(n-2)! + (n-3)!].$$

QUESTION 8.5. *The identity (24) holds for all $d = 2, \ldots, k$. Is there a combinatorial explanation?*

QUESTION 8.6. *For fixed k and n, what do the positive integers $g_n(k,d)$ count? Is there a combinatorial interpretation for $\beta_n(k) = \sum_{j=d}^{\min(n,k)} g_n(k,d)$, the multiplicity of the trivial representation in the top homology of $A_{n,k}^*$, in the nonstable case $k > n$? Recall that for $k \leqslant n$ this is the number $B_k^{\geqslant 2}$ of set partitions of $[k]$ with no singleton blocks, and is sequence OEIS A000296.*

QUESTION 8.7. *Recall that $a_{n-1}(n) = \binom{n-1}{2}$. Is there a combinatorial interpretation for the signed integers $a_i(n)$? There are many interpretations for $(-1)^{n-1}a_1(n) = (n-2)! + (n-3)!$, see OEIS A001048. For $n \geqslant 4$ it is the size of the largest conjugacy class in S_{n-1}. We were unable to find the other sequences $\{a_i(n)\}_{n \geqslant 3}$ in OEIS.*

9. THE SUBPOSET OF NORMAL WORDS

Let $N_{n,k}$ denote the poset of normal words of length at most k in $A_{n,k}^*$, again with an artificial top element $\hat{1}$ appended. Farmer showed that

THEOREM 9.1 (Farmer [9]). $\mu(N_{n,k}) = (-1)^{k-1}(n-1)^k = \mu(A_{n,k}^*)$, and $A_{n,k}, N_{n,k}$ both have the homology of a wedge of $(n-1)^k$ spheres of dimension $(k-1)$.

Björner and Wachs [7] showed that $N_{n,k}$ is dual CL-shellable and hence homotopy Cohen–Macaulay; it is therefore homotopy-equivalent to a wedge of $(n-1)^k$ spheres of dimension $(k-1)$. The order complexes of the posets $A_{n,k}^*$ and $N_{n,k}$ are thus homotopy-equivalent.

Using Quillen's fibre theorem ([16]) we can establish a slightly stronger result:

LEMMA 9.2. *Let $\alpha \in N_{n,k}^*$. Then the intervals $(\hat{0}, \alpha)_{N_{n,k}}$ and $(\hat{0}, \alpha)_{A_{n,k}^*}$ are $\mathrm{stab}(\alpha)$-homotopy equivalent, for the stabiliser subgroup $\mathrm{stab}(\alpha)$ of α. In particular the homology groups are all $\mathrm{stab}(\alpha)$-isomorphic.*

If $\alpha \in A_{n,k}^*$, but $\alpha \notin N_{n,k}^*$, then we know that the interval $(\hat{0}, \alpha)_{A_{n,k}^*}$ is contractible.

Proof. Let J_m be the set of words of length m that are *not* normal. Let

$$B_j = (\hat{0}, \alpha)_{A_{n,k}^*} \backslash (\cup_{m=j}^k J_m)$$

be the subposet obtained by removing all normal words at rank j and higher. Thus $B_1 = (\hat{0}, \alpha)_{N_{n,k}}$. Set $B_{k+1} = (\hat{0}, \alpha)_{A_{n,k}^*}$. We claim that the inclusion maps

$$(28) \qquad (\hat{0}, \alpha)_{N_{n,k}} = B_1 \subset B_2 \subset \cdots \subset B_j \subset B_{j+1} \subset B_{k+1} = (\hat{0}, \alpha)_{A_{n,k}^*}$$

are group equivariant homotopy equivalences. Note that

$$B_j = B_{j+1} \backslash \{\text{non-normal words of length } j+1\},$$

and B_j coincides with B_{j+1} for the first j ranks. The fibres to be checked are $F_{\leqslant w} = \{\beta \in B_j : \beta \leqslant \alpha\}$, for $w \in B_{j+1}$. If w is a normal word in B_{j+1}, then $w \in B_j$ and the fibre is the half-closed interval $(\hat{0}, w]$ in B_j; it is therefore contractible. If $w \in B_{j+1}$ is not a normal word, then $w \notin B_j$ and the interval $(\hat{0}, w)_{B_j}$ coincides with the same interval in $A_{n,k}^*$, so by Part (1) of Proposition 7.4, it is contractible. Hence by Quillen's fibre theorem, the inclusion induces a homotopy equivalence. $\square$

PROPOSITION 9.3. *The Whitney homology modules of $A_{n,k}^*$ and $N_{n,k}$ are S_n-isomorphic. In particular the jth Whitney homology of $N_{n,k}$ is isomorphic to the S_n-action on the elements at rank j.*

This statement is false for the dual Whitney homology. For instance,

$$\mu(ab, abab)_{A_{n,k}^*} = +3, \text{ but } \mu(ab, abab)_{N_{n,k}} = +1.$$

The first interval is a rank-one poset with four elements bab, aab, abb, aba, whereas the second consists only of two elements aba, bab.

It is also easy to find examples showing that the rank-selected homology is not the same for each poset.

EXAMPLE 9.4. Let $n = 2$ and consider the rank-set $\{1, 3\}$ for the poset $A_{2,k}^*$ and for its subposet of normal words $N_{2,k}$.

The words of length 3, all of which cover the two rank 1 elements a and b, are

$$aaa, aab, aba, abb, baa, bab, bba, bbb.$$

It is clear that the Möbius function values are

$$\mu(\hat{0}, aaa) = 0 = \mu(\hat{0}, bbb), \mu(\hat{0}, w) = -1 \text{ for all } w \notin \{aaa, bbb\}.$$

Hence the Möbius number of the rank-selected subposet of all words is -5, and from Theorem 4.2 the S_2-representation on homology is $3\, S_{(2)} \oplus 2\, S_{(1,1)}$. The order complex is a wedge of 5 one-dimensional spheres.

Now consider the corresponding rank-selected subposet of normal words: there are only two normal words of length 3, namely aba, bab and hence the Möbius number of the rank-selected subposet of normal words is -1, with trivial homology representation. The order complex is a one-dimensional sphere.

EXAMPLE 9.5. More generally, let S be the rank-set $[2, k]$, and consider the posets $A_{2,k}(S)$ and $N_{2,k}(S)$ obtained by deleting the atoms. Then by Theorem 4.2 the homology of $A_{2,k}(S)$ is $(k-1)S_{(1^2)}^{\otimes k} + kS_{(2)}$, while the homology of the normal word subposet $N_{2,k}(S)$ is seen to be $S_{(1^2)}^{\otimes k}$, which is either the trivial or the sign module, depending on the parity of k.

REMARK 9.6. In fact it is easy to see that $N_{2,k}$ is the ordinal sum [18] of k copies of an antichain of size 2, with a bottom and top element attached. Hence for any subset T of $[1, k]$, there is an S_2-equivariant poset isomorphism between $N_{2,k}(T)$ and $N_{2,|T|}$. Since the S_n-homology of $N_{2,k}$ is easily seen to be the k-fold tensor power of the sign representation, this determines $\tilde{H}(N_{2,k}(T))$ for all rank-sets T. Also, S_2 acts on the chains of any rank-selected subposet $N_{2,k}(T)$ like $2^{|T|-1}$ copies of the regular representation of S_2, since the 2^k chains of $N_{2,k}$ break up into orbits of size 2.

Recall from [18] that a finite graded poset P with $\hat{0}$ and $\hat{1}$ is Eulerian if its Möbius function μ_P satisfies $\mu(P) = (-1)^{\mathrm{rank}(y)-\mathrm{rank}(x)}$ for all intervals $(x, y) \subseteq (\hat{0}, \hat{1})$. It is known that all intervals $(x, y), y \neq \hat{1}$, in $N_{n,k}$ are Eulerian (see e.g. [18, Exercise 188]). In fact Björner and Wachs observed in [7] that for a finite alphabet $A = \{a_i : 1 \leqslant i \leqslant n\}$, the poset of normal words without the top element $N_{n,k}\backslash\{\hat{1}\}$ is simply Bruhat order on the Coxeter group with n generators a_i and relations $a_i^2 = 1$. Thus by lexicographic shellability [6], all intervals $(x, y), y \neq \hat{1}$ are homotopy equivalent to a sphere.

In fact this makes $N_{n,k}\backslash\{\hat{1}\}$ a CW-poset as defined in [4], and hence its order complex is isomorphic to the face poset of a regular CW-complex $\mathcal{K}$ [4, Proposition 3.1]. It is easy to see from the definitions that if a finite group G acts on a CW-poset P, then there is a G-module isomorphism between the jth Whitney homology $WH_j(P)$ and the G-action on the $(j-1)$-cells of the associated CW-complex $\mathcal{K}(P)$, which are simply the elements of P at rank j. This provides another explanation for the observation of Proposition 9.3.

An EL-labelling of the dual poset of normal (or Smirnov) words appears in [12]. In [13] the program of the present paper is carried out for the subposet of Smirnov words.

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