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Domino tilings and flips in dimensions 4 and higher

Caroline J. Klivans & Nicolau C. Saldanha

Abstract In this paper we consider domino tilings of bounded regions in dimension $n \geqslant 4$. We define the *twist* of such a tiling, an elements of $\mathbb{Z}/(2)$, and prove that it is invariant under flips, a simple local move in the space of tilings.

We investigate which regions $\mathcal{D}$ are *regular*, i.e. whenever two tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ of $\mathcal{D} \times [0, N]$ have the same twist then $\mathbf{t}_0$ and $\mathbf{t}_1$ can be joined by a sequence of flips provided some extra vertical space is allowed. We prove that all boxes are regular *except* $\mathcal{D} = [0, 2]^3$.

Furthermore, given a regular region $\mathcal{D}$, we show that there exists a value M (depending only on $\mathcal{D}$) such that if $\mathbf{t}_0$ and $\mathbf{t}_1$ are tilings of equal twist of $\mathcal{D} \times [0, N]$ then the corresponding tilings can be joined by a finite sequence of flips in $\mathcal{D} \times [0, N + M]$. As a corollary we deduce that, for regular $\mathcal{D}$ and large N, the set of tilings of $\mathcal{D} \times [0, N]$ has two twin giant components under flips, one for each value of the twist.

1. Introduction

A domino in dimension n is a $2 \times \overbrace{1 \times \cdots \times 1}^{n-1}$ rectangular block. We consider domino tilings of bounded cubiculated regions in $\mathbb{R}^n$ for $n \geqslant 4$. The case $n = 2$ has been extensively studied, with many remarkable results, see e.g. [1, 6, 16]. Almost every question about domino tilings seems to be much harder for $n \geqslant 3$, see e.g. [10].

The three dimensional case has distinctive behavior. The series of papers [4,8,9,12] investigate spaces of three-dimensional tilings, connectivity under local moves, and connections to certain algebraic parameters. Briefly summarizing:

A *flip* is a local move – two neighboring parallel dominoes are removed and placed back in a different position. Define an equivalence relation on the set $\mathcal{T}(\mathcal{R})$ of domino tilings of a region $\mathcal{R}$ as $\mathbf{t}_0 \approx \mathbf{t}_1$ if and only if the tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ can be joined by a finite number of flips.

- If a region $\mathcal{R}$ of dimension $n = 2$ is connected and simply connected then the equivalence relation is trivial: for any two tilings $\mathbf{t}_0, \mathbf{t}_1$ of the region $\mathcal{R}$ we have $\mathbf{t}_0 \approx \mathbf{t}_1$ (see [16]).
- Also for $n = 2$, if a region is planar and connected but not simply connected then the *flux* is an invariant under flips: $\mathbf{t}_0 \approx \mathbf{t}_1$ if and only if $\mathrm{Flux}(\mathbf{t}_0) = \mathrm{Flux}(\mathbf{t}_1)$. More generally, if $\mathcal{R}$ is a quadriculated surface then $\mathrm{Flux}(\mathbf{t}_0) \neq \mathrm{Flux}(\mathbf{t}_1)$ implies $\mathbf{t}_0 \not\approx \mathbf{t}_1$; if $\mathrm{Flux}(\mathbf{t}_0) = \mathrm{Flux}(\mathbf{t}_1)$, we usually (but not always) have $\mathbf{t}_0 \approx \mathbf{t}_1$ (see [14]).

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- For $n = 2$, if a region is planar and connected, the number of tilings of the region can be enumerated efficiently. Kasteleyn matrices, in particular, provide a linear algebraic approach to the counting problem (see [5]).
- For $n = 3$, if a region is contractible then the *twist* is an integer-valued invariant under flips. Thus, if $\mathrm{Tw}(\mathbf{t}_0) \neq \mathrm{Tw}(\mathbf{t}_1)$ then $\mathbf{t}_0 \not\approx \mathbf{t}_1$; if $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$, we usually (but not always) have $\mathbf{t}_0 \approx \mathbf{t}_1$ (see [4, 8, 9]).
- If $\mathcal{R}$ is a cubiculated manifold of dimension 3, $\mathrm{Flux}(\mathbf{t}_0)$ is also invariant under flips. If $\mathbf{t}_0 \approx \mathbf{t}_1$ we have $\mathrm{Flux}(\mathbf{t}_0) = \mathrm{Flux}(\mathbf{t}_1)$ and $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$. If $\mathrm{Flux}(\mathbf{t}_0) = \mathrm{Flux}(\mathbf{t}_1)$ and $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$ we usually (but not always) have $\mathbf{t}_0 \approx \mathbf{t}_1$ (see [4]).

In this paper we investigate the above concepts for $n \geqslant 4$. There is a fundamental shift in dimensions 4 and higher. In Section 3 we define the *twist* of a tiling which is no longer an element of $\mathbb{Z}$ but is naturally an element of $\mathbb{Z}/(2)$. The definition of twist for $n \geqslant 4$ is in a sense simpler, see Lemma 3.7. In Theorem 3.4 we prove that the twist is invariant under flips.

Sections 4 and 5 are concerned with enumeration and construct Kasteleyn matrices for the four dimensional case. As in [12], we focus on *cylinders*: regions of the form $\mathcal{R}_N = \mathcal{D} \times [0, N]$ where $\mathcal{D} \subset \mathbb{R}^{n-1}$ is a balanced contractible region. When $\mathcal{D}$ is fixed and N goes to infinity, we prove that the set of tilings $\mathcal{T}(\mathcal{R}_N)$ is almost evenly split between tilings with twists 0 and 1 (see Examples 2.2, 2.3 and Corollary 5.4). This is in contrast to the three-dimensional case where it is believed that the twist is normally distributed. Our result implies, however, that also in dimension $n = 3$, ($\mathrm{Tw}(\mathbf{t})$ mod 2) is almost evenly split between 0 and 1.

If $\mathbf{t}_0$ and $\mathbf{t}_1$ are tilings of $\mathcal{R}_{N_0}$ and $\mathcal{R}_{N_1}$, respectively, then $\mathbf{t}_0$ and $\mathbf{t}_1$ can be concatenated to define a tiling $\mathbf{t}_0 * \mathbf{t}_1$ of $\mathcal{R}_{N_0+N_1}$. If M is even, the region $\mathcal{R}_M$ admits a simple tiling, the *vertical* tiling $\mathbf{t}_{\mathrm{vert},M}$, formed by dominoes of the form $s \times [k, k+2]$ where $s \subset \mathcal{D}$ is a unit cube and $k \in [0, M)$ is an even integer. Also following [12], we define a weaker equivalence relation: $\mathbf{t}_0 \sim \mathbf{t}_1$ if and only if there exists an even integer M such that $\mathbf{t}_0 * \mathbf{t}_{\mathrm{vert},M} \approx \mathbf{t}_1 * \mathbf{t}_{\mathrm{vert},M}$. Under this equivalence relation, concatenation defines the *domino group* $G_{\mathcal{D}}$.

Given $\mathcal{D}$, we consider the *domino complex*, a 2-complex $\mathcal{C}_{\mathcal{D}}$ with a base point $\mathbf{p}_\circ$. Tilings of $\mathcal{R}_N$ are interpreted as closed paths of length N in $\mathcal{C}_{\mathcal{D}}$, starting and ending at $\mathbf{p}_\circ$. Two tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ satisfy $\mathbf{t}_0 \sim \mathbf{t}_1$ if and only if their paths are homotopic. Thus, there exists a natural isomorphism $G_{\mathcal{D}} \simeq \pi_1(\mathcal{C}_{\mathcal{D}}; \mathbf{p}_\circ)$ between the domino group and the fundamental group of $\mathcal{C}_{\mathcal{D}}$, see Sections 6 and 7.

For $n \geqslant 4$, a region $\mathcal{D} \subset \mathbb{R}^{n-1}$ is *regular* if and only if its domino group satisfies $G_{\mathcal{D}} \simeq \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$. Equivalently, $\mathcal{D}$ is regular if and only if $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$ implies $\mathbf{t}_0 \sim \mathbf{t}_1$ (where $\mathbf{t}_0$ and $\mathbf{t}_1$ are tilings of $\mathcal{R}_N = \mathcal{D} \times [0, N]$). For $n = 3$, a region $\mathcal{D} \subset \mathbb{R}^{n-1}$ is *regular* if and only if $G_{\mathcal{D}} \simeq \mathbb{Z} \oplus \mathbb{Z}/(2)$.

In Sections 8, 9, 10 and 11, we characterize the case for boxes as follows:

Theorem 1.1. *Let $n \geqslant 4$ and consider positive integers $L_1 \geqslant \cdots \geqslant L_{n-1} \geqslant 2$, at least one of them even. Consider the box $\mathcal{D} = [0, L_1] \times \cdots \times [0, L_{n-1}]$. If $n = 4$ and $L_1 = L_2 = L_3 = 2$ then $\mathcal{D}$ is not regular. In every other case, $\mathcal{D}$ is regular.*

Remark 1.2. In the irregular case $\mathcal{D} = [0, 2]^3 \subset \mathbb{R}^3$ there exists an isomorphism $G_{\mathcal{D}} \approx \mathbb{Z} \oplus \mathbb{Z}/(2)$. This example is discussed in Example 2.2 (Section 2) and the claim is proved in Lemma 8.3 (Section 8). The box $\mathcal{D} = [0, 2]^2 \times [0, 3] \subset \mathbb{R}^3$, on the other hand, is regular: see Remark 2.1, Example 2.3 and Lemma 9.1.

By definition, if $\mathcal{D}$ is regular and $\mathbf{t}_0, \mathbf{t}_1$ are tilings of $\mathcal{R}_N$ with $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1)$ then $\mathbf{t}_0$ and $\mathbf{t}_1$ can be joined by a finite sequence of flips provided some extra vertical space M is allowed. The next result, proved in Section 12, shows that the amount of extra space is bounded (as a function of N).

Theorem 1.3. *Let $n \geqslant 4$. If $\mathcal{D}$ is regular then there exists $M \in 2\mathbb{N}^* = \{2,4,6,\ldots\}$ (depending on $\mathcal{D}$ only) such that: if N is a positive integer and $\mathbf{t}_0, \mathbf{t}_1$ are tilings of $\mathcal{R}_N = \mathcal{D} \times [0,N]$ with $\operatorname{Tw}(\mathbf{t}_0) = \operatorname{Tw}(\mathbf{t}_1)$ then $\mathbf{t}_0 * \mathbf{t}_{\text{vert},M} \approx \mathbf{t}_1 * \mathbf{t}_{\text{vert},M}$.*

Corollary 1.4. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$ be a regular region, with $n \geqslant 4$. There exist connected components (under $\approx$) $T_i \subset \mathcal{T}(\mathcal{R}_N)$, $i \in \mathbb{Z}/(2)$, such that*

$$\lim_{N\to\infty} \frac{|T_0|}{|\mathcal{T}(\mathcal{R}_N)|} = \lim_{N\to\infty} \frac{|T_1|}{|\mathcal{T}(\mathcal{R}_N)|} = \frac{1}{2}, \qquad \limsup_{N\to\infty} \frac{\log|\mathcal{T}(\mathcal{R}_N) \smallsetminus (T_0 \cup T_1)|}{\log|\mathcal{T}(\mathcal{R}_N)|} < 1.$$

In other words, the set of tilings of $\mathcal{R}_N$ has two twin giant components. There are small components, but their total relative measure goes to zero exponentially (when $N \to \infty$).

As in the three dimensional case, it would be interesting to clarify which other contractible regions (not boxes!) are regular. It would also be interesting to study the domino group in other higher dimensional examples. We remind the reader that in the three dimensional case the domino group may have exponential growth. For instance, if $\mathcal{D} = [0,2] \times [0,L]$, $L \geqslant 3$, we construct in [12] a surjective homomorphism from $G^+_{\mathcal{D}}$ (a subgroup of index two of $G_{\mathcal{D}}$) to the free group F_2. Does something similar happen in higher dimensions? Notice that exponential growth of the domino group implies that all connected components under flips are small (unlike the situation described in Corollary 1.4).

2. Examples

In this section we present a few small examples. In all but the smallest cases, the results were obtained by computer; only some very small examples can be worked out by hand. We also show how to draw a tiling $\mathbf{t}$ of a region $\mathcal{R} \subset \mathbb{R}^4$, particularly if $\mathcal{R}$ is of the form $\mathcal{R} = \mathcal{R}_N = \mathcal{D} \times [0,N]$, $\mathcal{D} \subset \mathbb{R}^3$.

We first recall how tilings of $\mathcal{R}_N = \mathcal{D} \times [0,N] \subset \mathbb{R}^3$ are drawn in [12] for $\mathcal{D} \subset \mathbb{R}^2$, $\mathcal{D}$ a quadriculated disk. An example is given in Figure 1 for $\mathcal{D} = [0,3]^2$, $\mathcal{R} = \mathcal{D} \times [0,2]$. This region admits 229 tilings. The first and last tiling in Figure 1 admit no flip; the other 227 tilings form a single connected component under flips. A tiling is represented as a sequence of floors; vertical dominos (i.e. dominoes not contained in a floor) are represented by two squares, one in each floor.

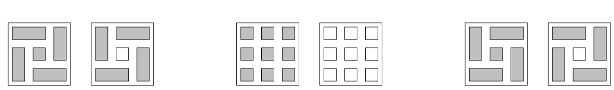

Figure 1. Three tilings of the box $\mathcal{R} = [0,3]^2 \times [0,2] \subset \mathbb{R}^3$.

Figures 2 and 4 show three and two tilings of the box $\mathcal{R} = [0,3]^2 \times [0,2]^2 \subset \mathbb{R}^4$, respectively; Figure 3 shows two tilings of $\mathcal{R} = [0,2]^4$. Let $x_1, \ldots, x_4$ be the coordinates of $\mathbb{R}^4$. Each 3×3 square in Figure 2 represents a slice of the form $i - 1 \leqslant x_3 \leqslant i$, $j - 1 \leqslant x_4 \leqslant j$, $i,j \in \{1,2\}$. The four squares (slices) are shown in the natural positions: $i = 1$ in the top row, $i = 2$ in the bottom row; $j = 1$ in the left column, $j = 2$ in the right column.

Dominoes in the directions x_1, x_2 are contained in slices and appear in the figure as dominoes. Dominoes in the directions x_3, x_4 appear as a pair of unit squares, one in one slice, one in another. A dark triangle in such unit squares indicates the position of the partner: it is as near the partner (in the figure) as possible. Thus, for instance,

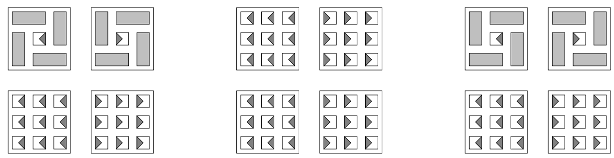

Figure 2. Three tilings of the box $\mathcal{R} = [0,3]^2 \times [0,2]^2 \subset \mathbb{R}^4$.

the two central unit squares in each 3×3 square in the top row of the first tiling in Figure 2 form a domino.

Remark 2.1. The first and third tilings in Figure 2 can be connected by a sequence of 22 flips. The reader should contrast this with the fact that the first and third tilings in Figure 1 can not be connected by a sequence of flips, not even if abundant extra 3-dimensional space with vertical dominoes is added around the box. Indeed, the two tilings $\mathbf{t}_{\pm 1}$ in Figure 1 have twists $\mathrm{Tw}(\mathbf{t}_1) = +1 \neq -1 = \mathrm{Tw}(\mathbf{t}_{-1})$ and flips preserve twist [4, 12].

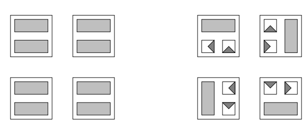

Figure 3. Two tilings of the box $\mathcal{R} = [0,2]^4 \subset \mathbb{R}^4$.

Example 2.2. The smallest non trivial region is $\mathcal{D} = [0,2]^3$. We describe the connected components via flips of the space of tilings of $\mathcal{R} = \mathcal{D} \times [0, N]$.

- For $N = 2$ there are 272 tilings and 9 components. The largest component has size 264 and includes all tilings of twist 0. There are 8 tilings of twist 1: each one is isolated, as no flip is possible. The first tiling in Figure 3 shows a tiling in the largest component; the second tiling is isolated.
- For $N = 3$ there are three components: the largest one has size 5985 (i.e. includes 5985 tilings) and twist 0; the other two have size 180 and twist 1.
- For $N = 4$ the components are: one of size 143065 and twist 0; two of size 6412 and twist 1; 56 components of sizes 1 or 2 and twist 0.
- For $N = 5$ the components are: one of size 3386376 and twist 0; two of size 202224 and twist 1; two of size 2028 and twist 0.
- For $N = 6$ the components are: one of size 80353593 and twist 0; two of size 5987060 and twist 1; two of size 98144 and twist 0; 392 components of sizes 1, 2 or 4 and twist 1.
- For $N = 30$ the approximate number of tilings of twist 0 and 1 are, respectively, $1.05 \cdot 10^{41}$ and $0.736 \cdot 10^{41}$.
- For $N = 50$ the approximate number of tilings of twist 0 and 1 are, respectively, $0.515 \cdot 10^{69}$ and $0.463 \cdot 10^{69}$.

As we shall prove in Lemma 8.1, the region $\mathcal{D}$ is not regular. This is consistent with the fact that there exist several large components in the space of tilings of $\mathcal{R}_N$ for large N.

Example 2.3. We now consider $\mathcal{D} = [0,2]^2 \times [0,3]$ and tilings of $\mathcal{R}_N = \mathcal{D} \times [0, N]$.

- For $N = 3$ the components are: one of size 762572 and twist 0 (T_0 in the notation of Corollary 1.4); one of size 99280 and twist 1 (T_1); 16 of size 16 and twist 0; 2 of size 2 and twist 0. Up to the obvious identification between $\mathcal{R}_3 = [0,2]^2 \times [0,3]^2$ and $[0,3]^2 \times [0,2]^2$, the tilings in Figures 2 and 4 are tilings of $\mathcal{R}_3$. The second tiling in Figure 2 belongs to T_0. The first and third tilings in Figure 2 both belong to T_1 (see Remark 2.1). Figure 4 shows two tilings in components of sizes 16 and 2.
- For $N = 4$ the components are: one of size 106303993 and twist 0 (T_0); one of size 20723112 and twist 1 (T_1); 8 of size 49 and twist 0; 16 of size 16 and twist 1; 16 of size 1 and twist 1.
- For $N = 30$ the approximate number of tilings of twist 0 and 1 are, respectively, $0.117 \cdot 10^{65}$ and $0.108 \cdot 10^{65}$.

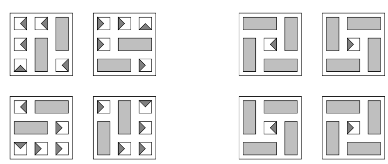

FIGURE 4. Two tilings of the box $\mathcal{R} = [0,3]^2 \times [0,2]^2 \subset \mathbb{R}^4$.

As we shall prove in Lemma 9.1, the box $\mathcal{D}$ is regular. This is consistent with the fact that there exist exactly two large components. For large N, the two large components have approximately the same size.

REMARK 2.4. The programs used to verify these examples were written in C or C++; sources available on the home page of the authors [7]. Most of them were written by the authors; some older programs were written by our collaborators from previous publications, including J. Freire and P. Milet, and by students, including B. Pereira. The cases $N \leqslant 6$ in Example 2.2 and the cases $N \leqslant 4$ in Example 2.3 were performed by brute force. Tilings are encoded by strings of characters: each character corresponds to a unit cube and indicates the direction of the corresponding domino. We first produced a list of all tilings in alphabetical order and then computed the connected components. The cases of larger N are too large for a brute force approach. We then use the theory explained in [13] and in Sections 3 and 4 below. In particular, in both examples we explicitly compute both the adjacency matrix A (as in Definition 4.1) and the matrix $\tilde{A}$ (as in Equation (6)). With the use of the arbitrary precision library gmp, this allows us to obtain the exact number of tilings with each value of the twist. Computing the sizes of the connected components appears to be significantly harder.

3. Twist

For the remainder of the paper, unless otherwise stated, by a region $\mathcal{R}$, we mean a balanced cubiculated subset of $\mathbb{R}^n$.

Given a region $\mathcal{R}$ let $\mathcal{T}(\mathcal{R})$ be the set of domino tilings of $\mathcal{R}$. Construct a simple bipartite graph $\mathcal{G}_{\mathcal{R}}$ as follows. Vertices of $\mathcal{G}_{\mathcal{R}}$ are unit cubes in $\mathcal{R}$ and two vertices of $\mathcal{G}_{\mathcal{R}}$ are joined by an edge if and only if the two corresponding unit cubes share a face of codimension one. We assume the vertices of $\mathcal{G}_{\mathcal{R}}$ belong to $\mathbb{Z}^n \subset \mathbb{R}^n$ and the edges of $\mathcal{G}_{\mathcal{R}}$ are unit segments. The color of a vertex $v = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ is given by $(-1)^{x_1+\cdots+x_n}$ (black is $+1$, white is -1). Let $b \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$ be the number of black vertices of $\mathcal{G}_{\mathcal{R}}$; recall that we assume that $\mathcal{R}$ is balanced so that $\mathcal{G}_{\mathcal{R}}$ also has b white vertices. Label the black vertices as $v_1, \ldots, v_b$ and the white vertices as

$w_1, \dots, w_b$. Tilings $\mathbf{t} \in \mathcal{T}(\mathcal{R})$ correspond to perfect matchings of $\mathcal{R}$, or, equivalently, to bijections $\sigma_{\mathbf{t}} : \{1, 2, \dots, b\} \to \{1, 2, \dots, b\}$ such that v_i is adjacent to $w_{\sigma(i)}$ (for all i).

The adjacency matrix of $\mathcal{R}$ is an indicator matrix recording if v_i and w_j are adjacent. Tilings of $\mathcal{R}$ naturally correspond to nonzero terms in the expansion of the determinant of the adjacency matrix. We imitate Kasteleyn's construction (originally for dimension 2 [5]) to define a matrix $K \in \mathbb{Z}^{b\times b}$ with entries $K_{ij} \in \{+1, -1\}$ if v_i and w_j are adjacent and $K_{ij} = 0$ otherwise. As above, consider $v_i, w_j \in \mathbb{Z}^n$ so that $v_i - w_j = \pm e_k$ for some $k \in \{1, 2, \dots, n\}$. Write $v_i = (x_1, \dots, x_n) \in \mathbb{Z}^n$ so that $w_j = (x_1, \dots, x_k \pm 1, \dots, x_n)$ and set

$$K_{ij} = (-1)^{x_1+\cdots+x_{k-1}}. \tag{1}$$

As in the case of the adjacency matrix, tilings of $\mathcal{R}$ naturally correspond to nonzero terms in the expansion of $\det(K)$. Given a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R})$, define the signed permutation matrix $T_{\mathbf{t}}$ by $(T_{\mathbf{t}})_{i,\sigma_{\mathbf{t}}(i)} = K_{i,\sigma_{\mathbf{t}}(i)} \in \{+1, -1\}$ and $(T_{\mathbf{t}})_{ij} = 0$ otherwise.

Definition 3.1. *The* twist $\mathrm{Tw}(\mathbf{t}) \in \mathbb{Z}/(2)$ *is defined by*

$$\det(T_{\mathbf{t}}) = (-1)^{\mathrm{Tw}(\mathbf{t})} = \mathrm{sign}(\sigma_{\mathbf{t}}) \prod_i K_{i,\sigma_{\mathbf{t}}(i)}. \tag{2}$$

Definition 3.2. *The* defect $\Delta(\mathcal{R})$ *of a region* $\mathcal{R}$*:*

$$\begin{aligned} \Delta(\mathcal{R}) = \det(K) &= \sum_{\mathbf{t}\in\mathcal{T}(\mathcal{R})} (-1)^{\mathrm{Tw}(\mathbf{t})} \\ &= |\{\mathbf{t} \in \mathcal{T}(\mathcal{R}) \mid \mathrm{Tw}(\mathbf{t}) = 0\}| - |\{\mathbf{t} \in \mathcal{T}(\mathcal{R}) \mid \mathrm{Tw}(\mathbf{t}) = 1\}|. \end{aligned} \tag{3}$$

Here $\mathrm{sign}(\sigma) = (-1)^{\mathrm{inv}(\sigma)}$ *is the sign of the permutation* σ*,* $\mathrm{inv}(\sigma) = |\mathrm{Inv}(\sigma)|$ *is the number of inversions of* σ *and* $\mathrm{Inv}(\sigma)$ *is the set of inversions of* σ*.*

Therefore the determinant of our Kasteleyn matrix does not enumerate the total number of tilings. Instead it detects the difference between the number of tilings with twist 0 and twist 1.

Remark 3.3. The definition of Tw depends on the labeling of the vertices of $\mathcal{G}_{\mathcal{R}}$. Changing labelings corresponds to permuting rows and columns of K and therefore possibly changing the value of $\mathrm{Tw}(\mathbf{t}) \in \mathbb{Z}/(2)$ for all tilings $\mathbf{t}$.

A *flip* is a local move: remove two adjacent parallel dominoes and place them back in the only other possible way. A *trit* is another local move. Consider a block formed by 8 unit cubes, of dimensions $2 \times 2 \times 2 \times 1 \times \cdots \times 1$: if we remove from the block two opposite unit cubes, we are left with the union of six unit cubes, which can be tiled by dominoes in precisely two ways. A trit consists in finding three dominoes in the configuration above, removing them and placing them back in the only other possible way. The trit is therefore a local move involving three dominoes. All other local moves involving three (or fewer) dominoes reduce to a (very short) sequence of flips, as can be verified case by case. The trit does not reduce to a sequence of flips, as shown in the next theorem.

Theorem 3.4. *Let* $\mathbf{t}_0$ *and* $\mathbf{t}_1$ *be tilings of a region* $\mathcal{R}$*. If* $\mathbf{t}_0$ *and* $\mathbf{t}_1$ *differ by a flip then* $\mathrm{Tw}(\mathbf{t}_1) = \mathrm{Tw}(\mathbf{t}_0)$*. If* $\mathbf{t}_0$ *and* $\mathbf{t}_1$ *differ by a trit then* $\mathrm{Tw}(\mathbf{t}_1) = 1 - \mathrm{Tw}(\mathbf{t}_0)$*.*

Proof. A flip is always contained in a plane, an affine subspace of dimension 2. Assume the two relevant dimensions are k_0 and k_1 with $1 \leqslant k_0 < k_1 \leqslant n$. Among the four vertices involved, let $v = (x_1, \dots, x_n)$ be the one with smallest coordinates so that the other three vertices are $v + e_{k_0}$, $v + e_{k_1}$ and $v + e_{k_0} + e_{k_1}$. Without loss of generality, suppose $\mathbf{t}_0$ contains the dominoes $(v, v + e_{k_0})$ and $(v + e_{k_1}, v + e_{k_0} + e_{k_1})$ and $\mathbf{t}_1$ contains the dominoes $(v, v + e_{k_1})$ and $(v + e_{k_0}, v + e_{k_0} + e_{k_1})$.

Each tiling defines a bijection from the set of black vertices to the set of white vertices. The two bijections corresponding to $\mathbf{t}_0$ and $\mathbf{t}_1$ differ by a transposition (on either side) and therefore have opposite parities. Also, the matrix K assigns signs to edges. The signs assigned to $(v, v+e_{k_0})$ and $(v+e_{k_1}, v+e_{k_0}+e_{k_1})$ both equal $(-1)^{x_1+\cdots+x_{k_0-1}}$ and are therefore equal. The signs assigned to $(v, v+e_{k_1})$ and $(v+e_{k_0}, v+e_{k_0}+e_{k_1})$ are $(-1)^{x_1+\cdots+x_{k_0}+\cdots+x_{k_1-1}}$ and $(-1)^{x_1+\cdots+(x_{k_0}+1)+\cdots+x_{k_1-1}}$ and are therefore different. We thus have $\det(T_{\mathbf{t}_1}) = \det(T_{\mathbf{t}_0})$ and therefore $\mathrm{Tw}(\mathbf{t}_1) = \mathrm{Tw}(\mathbf{t}_0)$, proving the first claim.

A trit is always contained in an affine subspace of dimension 3. Assume the relevant dimensions to be k_0, k_1, k_2 with $1 \leqslant k_0 < k_1 < k_2 \leqslant n$. Assume that the three dominoes $(v+e_{k_0}, v+e_{k_0}+e_{k_1})$, $(v+e_{k_1}, v+e_{k_1}+e_{k_2})$ and $(v+e_{k_2}, v+e_{k_2}+e_{k_0})$ are contained in $\mathbf{t}_0$ and that the three dominoes $(v+e_{k_0}, v+e_{k_0}+e_{k_2})$, $(v+e_{k_1}, v+e_{k_1}+e_{k_0})$ and $(v+e_{k_2}, v+e_{k_2}+e_{k_1})$ are contained in $\mathbf{t}_1$. As above, write $v = (x_1, \ldots, x_n)$.

The bijections corresponding to $\mathbf{t}_0$ and $\mathbf{t}_1$ now differ by a 3-cycle and therefore have the same parity. The signs assigned by K to $(v+e_{k_2}, v+e_{k_2}+e_{k_0})$ and $(v+e_{k_1}, v+e_{k_1}+e_{k_0})$ are both $(-1)^{x_1+\cdots+x_{k_0-1}}$ and therefore equal. The signs assigned to $(v+e_{k_2}, v+e_{k_2}+e_{k_1})$ and $(v+e_{k_0}, v+e_{k_0}+e_{k_1})$ are $(-1)^{x_1+\cdots+x_{k_0}+\cdots+x_{k_1-1}}$ and $(-1)^{x_1+\cdots+(x_{k_0}+1)+\cdots+x_{k_1-1}}$, respectively, and therefore different. Finally, the signs assigned to $(v+e_{k_1}, v+e_{k_1}+e_{k_2})$ and $(v+e_{k_0}, v+e_{k_0}+e_{k_2})$ are

$$(-1)^{x_1+\cdots+x_{k_0}+\cdots+(x_{k_1}+1)+\cdots+x_{k_2-1}}, \quad (-1)^{x_1+\cdots+(x_{k_0}+1)+\cdots+x_{k_1}+\cdots+x_{k_2-1}},$$

respectively, and therefore equal. We thus have $\det(T_{\mathbf{t}_1}) = -\det(T_{\mathbf{t}_0})$ and therefore $\mathrm{Tw}(\mathbf{t}_1) = 1 - \mathrm{Tw}(\mathbf{t}_0)$, proving the second claim. □

Before moving on, we show the naturality of the matrix K.

Consider a region $\mathcal{R} \subset \mathbb{R}^n$ and its graph $\mathcal{G}_{\mathcal{R}}$. A *Kasteleyn system* for $\mathcal{R}$ assigns to each edge of $\mathcal{G}_{\mathcal{R}}$ a coefficient $+1$ or -1 satisfying the following condition: if four edges form a square then the product of their coefficients is -1. Given a Kasteleyn system we also have a *Kasteleyn matrix* for $\mathcal{R}$, a matrix $\tilde{K} \in \mathbb{Z}^{b\times b}$: if v_i and w_j are adjacent then $\tilde{K}_{ij}$ is the coefficient of the edge $v_i w_j$ (and the coefficients form a Kasteleyn system). Thus, for all i, j, $\tilde{K}_{ij} \in \{+1, -1\}$ if and only if v_i and w_j are adjacent. Also, for all i_0, i_1, j_0, j_1, we have $\tilde{K}_{i_0 j_0}\tilde{K}_{i_0 j_1}\tilde{K}_{i_1 j_0}\tilde{K}_{i_1 j_1} \in \{0, -1\}$.

The matrix K is an example of a Kasteleyn matrix. The following lemma shows that if $\mathcal{R}$ is connected and simply connected then any other Kasteleyn matrices are only minor variations.

Lemma 3.5. *Consider a region $\mathcal{R} \subset \mathbb{R}^n$. Assume furthermore that $\mathcal{R}$ is connected and simply connected. Then $\tilde{K}$ is a Kasteleyn matrix if and only if there exist diagonal matrices $D_{\mathrm{bl}}, D_{\mathrm{wh}}$ with diagonal entries equal to ± 1 and $\tilde{K} = D_{\mathrm{bl}} K D_{\mathrm{wh}}$.*

Proof. It is straightforward to verify that if $D_{\mathrm{bl}}, D_{\mathrm{wh}}$ are diagonal matrices with diagonal entries equal to ± 1 then $D_{\mathrm{bl}} K D_{\mathrm{wh}}$ is indeed a Kasteleyn matrix. In order to prove the converse we use the language of homology. Consider a Kasteleyn matrix $\tilde{K}$ and its corresponding Kasteleyn system. A Kasteleyn system defines an element of $\alpha \in C^1(\mathcal{R}; \mathbb{Z}/(2))$: if e is an edge then the coefficient of e in the Kasteleyn system is $(-1)^{\alpha(e)}$, $\alpha(e) \in \mathbb{Z}/(2)$. (Here $C^1(\mathcal{R}; \mathbb{Z}/(2))$ is the first cochain group of the cell complex $\mathcal{R}$ with coefficients in $\mathbb{Z}/(2)$.) Let $\alpha, \tilde{\alpha} \in C^1(\mathcal{R}; \mathbb{Z}/(2))$ correspond to the original K and to $\tilde{K}$, respectively. By definition, if s is an oriented square then $\alpha(\partial s) = \tilde{\alpha}(\partial s) = 1 \in \mathbb{Z}/(2)$. (Here $\partial : C_2 \to C_1$ is the boundary map.) Thus $(\alpha - \tilde{\alpha})(\partial s) = 0$ for all s and $\alpha - \tilde{\alpha} \in Z^1$ (i.e. it is closed). Since $\mathcal{R}$ is simply connected we have from the universal coefficient theorem that $H^1(\mathcal{R}; \mathbb{Z}/(2)) = 0$: it follows that $\alpha - \tilde{\alpha} \in B^1$ (i.e. it is exact). (Here $B^1 \subseteq C^1$ is the image of the coboundary map $\partial^* : C^0 \to C^1$.) In other words, there exists $\delta \in C^0(\mathcal{R}; \mathbb{Z}/(2))$ with $\alpha - \tilde{\alpha} = \partial^* \delta$. For any edge $e = vw$

we have $\alpha(e) - \tilde{\alpha}(e) = \delta(w) - \delta(v)$. Thus, δ gives us the desired diagonal matrices D_{bl} and D_{wh}. □

Corollary 3.6. *Consider a connected and simply connected region $\mathcal{R} \subset \mathbb{R}^n$, $n \geqslant 3$. Consider a fixed Kasteleyn matrix $\tilde{K}$ for $\mathcal{R}$. For a tiling $\mathbf{t}$ of $\mathcal{R}$, construct a signed permutation matrix $\tilde{T} = T_{\mathbf{t},\tilde{K}}$ with nonzero entries $\tilde{T}_{ij} = \tilde{K}_{ij}$ when v_i and w_j form a domino of $\mathbf{t}$. Then there exists $\varepsilon \in \{+1, -1\}$ such that for all $\mathbf{t}$ we have $\det(T_{\mathbf{t},\tilde{K}}) = \varepsilon \operatorname{Tw}(\mathbf{t})$.*

Proof. By construction, $\operatorname{Tw}(\mathbf{t}) = \det(T_{\mathbf{t},K})$ for the original Kasteleyn matrix K. From Lemma 3.5, there exist diagonal matrices D_{bl} and D_{wh} with $\tilde{K} = D_{\mathrm{bl}} K D_{\mathrm{wh}}$. By construction we also have $T_{\mathbf{t},\tilde{K}} = D_{\mathrm{bl}} T_{\mathbf{t},K} D_{\mathrm{wh}}$. Take $\varepsilon = \det(D_{\mathrm{bl}} D_{\mathrm{wh}})$: we have $\det(T_{\mathbf{t},\tilde{K}}) = \varepsilon \det(T_{\mathbf{t},K})$ for all $\mathbf{t}$, as desired. □

In dimension $n = 3$, the twist $\operatorname{Tw}(\mathbf{t})$ is defined to be an *integer* (see [8], [4]). In order to avoid confusion, we temporarily write, for $n = 3$, $\operatorname{Tw}_{\mathbb{Z}}$ for the twist as defined in the other references and $\operatorname{Tw}_{\mathbb{Z}/(2)}$ for the twist as defined here. The following lemma clarifies the relationship between the two concepts.

Lemma 3.7. *Let $\mathcal{D} \subset \mathbb{R}^2$ be a balanced quadriculated disk. Let $\mathcal{R}_N = \mathcal{D} \times [0, N] \subset \mathbb{R}^3$. For any tiling $\mathbf{t}$ of $\mathcal{R}_N$ we have $\operatorname{Tw}_{\mathbb{Z}/(2)}(\mathbf{t}) = (\operatorname{Tw}_{\mathbb{Z}}(\mathbf{t}) \bmod 2)$.*

The proof below relies heavily on notation, definitions and results from [11] and [4]. We feel that providing a more self-contained exposition would imply too much repetition.

Proof. Here, $\operatorname{Tw}_{\mathbb{Z}}$ is given by Definition 7.7 from [4]. Since $\mathcal{R}_N$ is contractible the flux is 0 and $m = 0$. Given two tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ which differ by a cycle, Definition 7.2 gives us $\operatorname{Tw}_{\mathbb{Z}}(\mathbf{t}_1) - \operatorname{Tw}_{\mathbb{Z}}(\mathbf{t}_0) = \phi(t_1; t_1 - t_0)$. It thus suffices to check that $\operatorname{Tw}_{\mathbb{Z}/(2)}(\mathbf{t}_1) - \operatorname{Tw}_{\mathbb{Z}/(2)}(\mathbf{t}_0) = (\phi(t_1; t_1 - t_0) \bmod 2)$. If there exists a Seifert surface for the cycle $t_1 - t_0$ then this follows from Kasteleyn systems, as discussed in [11]. More generally, we may take refinements, as in [4]. □

4. Plugs and floors

For the remainder of the paper, all regions $\mathcal{D} \subset \mathbb{R}^{n-1}$ are assumed to be balanced, cubiculated and contractible.

Consider a region $\mathcal{D} \subset \mathbb{R}^{n-1}$: we are interested in tilings of $\mathcal{R}_N = \mathcal{D} \times [0, N]$. We imitate some of the constructions from [12], where the case $n = 3$ is discussed.

A domino d (of dimension n) contained in $\mathcal{R}_N$ is *horizontal* if it is of the form $\tilde{d} \times [k-1, k]$ where $\tilde{d} \subset \mathcal{D}$ is a domino (of dimension $n-1$). A domino $d \subset \mathcal{R}_N$ is *vertical* otherwise, i.e. if it is of the from $s \times [k-1, k+1]$ where $s \subset \mathcal{D}$ is a unit cube. A *plug* is a balanced set of unit cubes contained in $\mathcal{D}$ (or balanced set of vertices in $\mathcal{G}_{\mathcal{D}}$). This includes the empty plug $\mathbf{p}_\circ = \varnothing$ and its complement $\mathbf{p}_\bullet = \mathcal{D}$. Let $\mathcal{P} = \mathcal{P}_{\mathcal{D}}$ be the set of all plugs. Two plugs $p_0, p_1 \in \mathcal{P}$ are *disjoint* if and only if $p_0 \cap p_1 = \mathbf{p}_\circ$. A *floor* is a triple (p_0, f, p_1) where $p_0, p_1 \in \mathcal{P}$ are disjoint plugs and f is a domino tiling of $\mathcal{D}_{p_0,p_1} = \mathcal{D} \smallsetminus (p_0 \cup p_1)$. A tiling of $\mathcal{R}_N$ can be identified with a alternating sequence of plugs and floors:

$$\mathbf{t} = (p_0 = \mathbf{p}_\circ, \mathbf{f}_1, p_1, \dots, p_{N-1}, \mathbf{f}_N, p_N = \mathbf{p}_\circ). \tag{4}$$

Here p_k is the set of unit cubes $s \subset \mathcal{D}$ such that the vertical domino $s \times [k-1, k+1]$ is contained in $\mathbf{t}$. Also, $\mathbf{f}_k = (p_{k-1}, f_k, p_k)$ where f_k consists of dominoes $\tilde{d} \subset \mathcal{D}$ such that the horizontal domino $d \times [k-1, k]$ is contained in $\mathbf{t}$.

The *domino complex* $\mathcal{C}_{\mathcal{D}}$ is a 2-complex associated to $\mathcal{D}$. We first construct a graph $\mathcal{C}_{1,\mathcal{D}}$ which is essentially the 1-skeleton of $\mathcal{C}_{\mathcal{D}}$; the complex itself will be constructed in Section 6. The set of vertices of $\mathcal{C}_{1,\mathcal{D}}$ is the set of plugs $\mathcal{P}$. If $p_0, p_1 \in \mathcal{P}$ are not disjoint there is no edge joining them. If p_0 and p_1 are disjoint there is one edge for every tiling of $\mathcal{D}_{p_0,p_1}$. Thus, a floor $\mathbf{f}_1 = (p_0, f_1, p_1)$ is identified with an edge joining p_0 and p_1. Each tiling of $\mathcal{D} = \mathcal{D}_{\mathbf{p}_\circ,\mathbf{p}_\circ}$ yields a loop based on the vertex $\mathbf{p}_\circ$; these are the only loops in $\mathcal{C}_{1,\mathcal{D}}$.

Each tiling $\mathbf{t}$ of $\mathcal{R}_N$ is identified with a closed walk of length N in $\mathcal{C}_{1,\mathcal{D}}$ from $\mathbf{p}_\circ$ to itself. More generally, consider the *cork*

$$\mathcal{R}_{0,N;p_0,p_N} = \mathcal{R}_N \smallsetminus ((p_0 \times [0,1]) \cup (p_N \times [N-1,N])).$$

Each tiling of $\mathcal{R}_{0,N;p_0,p_N}$ is identified with a walk in $\mathcal{C}_{\mathcal{D}}$, of length N, starting at p_0 and ending at p_N. In order to count tilings, we construct the adjacency matrix A of $\mathcal{C}_{1,\mathcal{D}}$.

Definition 4.1. *The* adjacency matrix $A \in \mathbb{Z}^{\mathcal{P}\times\mathcal{P}}$ *of* $\mathcal{C}_{1,\mathcal{D}}$ *is the matrix given by:*

$$A_{p_0,p_1} = \begin{cases} |\mathcal{T}(\mathcal{D}_{p_0,p_1})|, & p_0 \cap p_1 = \mathbf{p}_\circ, \\ 0, & p_0 \cap p_1 \neq \mathbf{p}_\circ. \end{cases}$$

Thus, $(A^N)_{p_0,p_N}$ is the number of tilings of $\mathcal{R}_{0,N;p_0,p_N}$. In particular,

$$|\mathcal{T}(\mathcal{R}_N)| = (A^N)_{\mathbf{p}_\circ,\mathbf{p}_\circ}.$$

In order to compute the defect $\Delta(\mathcal{R}_N)$ we first define $\operatorname{tw}_{p_0,p_1}(\mathbf{t}) \in \mathbb{Z}/(2)$ for a tiling $\mathbf{t}$ of $\mathcal{D}_{p_0,p_1}$.

Label the unit cubes of $\mathcal{D}$: the black cubes are $v_1, \ldots, v_b$; the white cubes are $w_1, \ldots, w_b$. Construct a Kasteleyn matrix K for $\mathcal{D}$ as above. The plug $p_i \in \mathcal{P}$ consists of b_i black unit cubes and b_i white unit cubes, thus defining two subsets $P_{i,\mathrm{bl}}, P_{i,\mathrm{wh}} \subseteq \{1,\ldots,b\}$ with $|P_{i,\mathrm{bl}}| = |P_{i,\mathrm{wh}}| = b_i$: $j \in P_{i,\mathrm{bl}}$ if and only if v_j is contained in p_i (and similarly for white). If p_0 and p_1 are disjoint then $P_{0,\mathrm{bl}}$ and $P_{1,\mathrm{bl}}$ are disjoint and so are $P_{0,\mathrm{wh}}$ and $P_{1,\mathrm{wh}}$. Define subsets $D_{p_0,p_1,\mathrm{bl}}, D_{p_0,p_1,\mathrm{wh}} \subseteq \{1,\ldots,b\}$ and functions $h_{\mathrm{bl}}, h_{\mathrm{wh}} : \{1,\ldots,b\} \to \{0,\pm1\}$ by

$$D_{p_0,p_1,\mathrm{bl}} = \{1,\ldots,b\} \smallsetminus (P_{0,\mathrm{bl}} \cup P_{1,\mathrm{bl}}), \quad D_{p_0,p_1,\mathrm{wh}} = \{1,\ldots,b\} \smallsetminus (P_{0,\mathrm{wh}} \cup P_{1,\mathrm{wh}}),$$
$$h_{\mathrm{bl}}(i) = [i \in P_{1,\mathrm{bl}}] - [i \in P_{0,\mathrm{bl}}], \quad h_{\mathrm{wh}}(i) = [i \in P_{1,\mathrm{wh}}] - [i \in P_{0,\mathrm{wh}}],$$

so that, for instance, $i \in D_{p_0,p_1,\mathrm{bl}}$ if and only if $h_{\mathrm{bl}}(i) = 0$; we use here Iverson's notation. Define the subsets $\operatorname{Inv}_{\mathrm{bl}}, \operatorname{Inv}_{\mathrm{wh}} \subseteq \{1,\ldots,b\}^2$ and non negative integers $\operatorname{inv}_{\mathrm{bl},p_0,p_1}, \operatorname{inv}_{\mathrm{wh},p_0,p_1} \in \mathbb{N}$ by

$$\operatorname{Inv}_* = \{(i_0,i_1) \in \{1,\ldots,b\}^2 \mid i_0 < i_1, h_*(i_0) > h_*(i_1)\}, \quad \operatorname{inv}_{*,p_0,p_1} = |\operatorname{Inv}_*|.$$

A tiling $\mathbf{t} \in \mathcal{T}(\mathcal{D}_{p_0,p_1})$ is defined by a bijection $\sigma_{\mathbf{t}} : D_{p_0,p_1,\mathrm{bl}} \to D_{p_0,p_1,\mathrm{wh}}$ such that $K_{i,\sigma_{\mathbf{t}}(i)} \in \{+1,-1\}$ for all $i \in D_{p_0,p_1,\mathrm{bl}}$. Define the subset $\operatorname{Inv}(\sigma_{\mathbf{t}}) \subseteq D^2_{p_0,p_1,\mathrm{bl}}$ and the non negative integer $\operatorname{inv}(\sigma_{\mathbf{t}}) \in \mathbb{N}$ by

$$\operatorname{Inv}(\sigma_{\mathbf{t}}) = \{(i_0,i_1) \in D^2_{p_0,p_1,\mathrm{bl}} \mid i_0 < i_1, \sigma_{\mathbf{t}}(i_0) > \sigma_{\mathbf{t}}(i_1)\}, \quad \operatorname{inv}(\sigma_{\mathbf{t}}) = |\operatorname{Inv}(\sigma_{\mathbf{t}})|.$$

Finally, for $\mathbf{t} \in \mathcal{T}(\mathcal{D}_{p_0,p_1})$, define $\operatorname{tw}_{p_0,p_1}(\mathbf{t}), \operatorname{tk}(\mathbf{t}) \in \mathbb{Z}/(2)$ by

$$\operatorname{tw}_{p_0,p_1}(\mathbf{t}) = (\operatorname{tk}(\mathbf{t}) + \operatorname{inv}(\sigma_{\mathbf{t}}) + \operatorname{inv}_{\mathrm{bl},p_0,p_1} + \operatorname{inv}_{\mathrm{wh},p_0,p_1}) \bmod 2, \tag{5}$$
$$(-1)^{\operatorname{tk}(\mathbf{t})} = \prod_{i \in D_{p_0,p_1,\mathrm{bl}}} K_{i,\sigma_{\mathbf{t}}(i)}.$$

Let $\mathcal{D} \subset \mathbb{R}^{n-1}$. Let $\tilde{A} \in \mathbb{Z}^{\mathcal{P}\times\mathcal{P}}$ be defined by

$$\tilde{A}_{p_0,p_1} = \sum_{\mathbf{t} \in \mathcal{T}(\mathcal{D}_{p_0,p_1})} (-1)^{\operatorname{tw}_{p_0,p_1}(\mathbf{t})}, \tag{6}$$

if p_0 and p_1 are not disjoint then $\tilde{A}_{p_0,p_1} = 0$.

Lemma 4.2. *For $\tilde{A}$ as defined in Equation 6 and $N \in \mathbb{N}$,*

$$\Delta(\mathcal{R}_N) = (\tilde{A}^N)_{\mathbf{p}_\circ,\mathbf{p}_\circ}.$$

Proof. Write $\mathbf{t} \in \mathcal{T}(\mathcal{R}_N)$ as a sequence of plugs and floors, as in Equation 4. We claim that

$$\operatorname{Tw}(\mathbf{t}) = \sum_{1 \leqslant k \leqslant N} \operatorname{tw}_{p_{k-1},p_k}(f_k),$$

which completes the proof. For this, we go back to the definition of $\operatorname{Tw}(\mathbf{t})$ in Equation 2 and compute $\operatorname{inv}(\sigma_{\mathbf{t}})$ and $\kappa = \prod_i K_{i,\sigma_{\mathbf{t}}(i)}$. Let b_k be the number of black unit cubes in p_k: we claim that

$$\operatorname{inv}(\sigma_{\mathbf{t}}) - \sum_{1 \leqslant k \leqslant N} \left(\operatorname{inv}(\sigma_{f_k}) + \operatorname{inv}_{\mathrm{bl},p_{k-1},p_k} + \operatorname{inv}_{\mathrm{wh},p_{k-1},p_k}\right) = \sum_{1 \leqslant k < N} b_k^2. \tag{7}$$

Let b be the number of black unit cubes in $\mathcal{D}$. First label black and white unit cubes in $\mathcal{D}$. Next label unit cubes in $\mathcal{R}_N$, using the previous labels in each floor and proceeding by increasing floor. In particular, for $i \in \{1, \dots, Nb\}$, both the i-th black and white unit cubes are contained in floor $\lceil i/b \rceil \in \{1, \dots, N\}$. We therefore have $|\lceil i/b \rceil - \lceil \sigma_{\mathbf{t}}(i)/b \rceil| \leqslant 1$ for all i.

Recall that an inversion for $\sigma_{\mathbf{t}}$ is a pair (i_0, i_1) of indices for black unit cubes such that $i_0 < i_1$ and $j_0 = \sigma_{\mathbf{t}}(i_0) > \sigma_{\mathbf{t}}(i_1) = j_1$. We thus have $\lceil i_0/b \rceil \leqslant \lceil i_1/b \rceil$ and $\lceil j_0/b \rceil \geqslant \lceil j_1/b \rceil$. We consider the possible cases.

- If $\lceil i_0/b \rceil = \lceil i_1/b \rceil$ and $\lceil j_0/b \rceil = \lceil j_1/b \rceil$ we may write $k = \lceil i_0/b \rceil = \lceil j_0/b \rceil$. Both dominoes are then contained in floor k, and the inversion (i_0, i_1) is counted once by $\operatorname{inv}(\sigma_{\mathbf{t}})$ and once by $\operatorname{inv}(\sigma_{f_k})$. Strictly speaking, in the second case the inversion is now called $(i_0 - (k-1)b, i_1 - (k-1)b)$, but we shall not follow such relabelings from now on.
- If $k = \lceil i_0/b \rceil = \lceil i_1/b \rceil$ and $\lceil j_0/b \rceil > \lceil j_1/b \rceil$ then the inversion (i_0, i_1) is counted once by $\operatorname{inv}(\sigma_{\mathbf{t}})$ and once by $\operatorname{inv}_{\mathrm{bl},p_{k-1},p_k}$.
- If $k = \lceil j_0/b \rceil = \lceil j_1/b \rceil$ and $\lceil i_0/b \rceil < \lceil i_1/b \rceil$ then the inversion (i_0, i_1) is counted once by $\operatorname{inv}(\sigma_{\mathbf{t}})$ and once by $\operatorname{inv}_{\mathrm{wh},p_{k-1},p_k}$ (in the second case it is called $(j_1 - (k-1)b, j_0 - (k-1)b)$).
- Finally, if $k = \lceil i_0/b \rceil < \lceil i_1/b \rceil = k+1$ and $k = \lceil j_1/b \rceil < \lceil j_0/b \rceil = k+1$ then the inversion (i_0, i_1) is counted once by $\operatorname{inv}(\sigma_{\mathbf{t}})$ and not counted by the summation on the left hand side. For each k, there exist b_k^2 such inversions, completing the proof of Equation 7.

Write $\kappa = \kappa_{\mathrm{hz}}\kappa_{\mathrm{vt}}$, where

$$\kappa_{\mathrm{hz}} = \prod_{\lceil i/b \rceil = \lceil \sigma_{bt}(i)/b \rceil} K_{i,\sigma_{\mathbf{t}}(i)}, \qquad \kappa_{\mathrm{vt}} = \prod_{\lceil i/b \rceil \neq \lceil \sigma_{bt}(i)/b \rceil} K_{i,\sigma_{\mathbf{t}}(i)}.$$

For each k, $1 \leqslant k < N$, we have

$$\prod_{\{\lceil i/b \rceil, \lceil \sigma_{bt}(i)/b \rceil\} = \{k,k+1\}} K_{i,\sigma_{\mathbf{t}}(i)} = (-1)^{b_k}$$

and therefore $\kappa_{\mathrm{vt}} = (-1)^{(b_1 + \cdots + b_{N-1})}$. For $1 \leqslant k \leqslant N$, let

$$\kappa_k = (-1)^{\operatorname{tk}(f_k)} = \prod_{i \in D_{p_{k-1},p_k,\mathrm{bl}}} K_{i,\sigma_{f_k}(i)}$$

so that $\kappa_{\mathrm{hz}} = \prod_k \kappa_k$ and therefore $\kappa \cdot (\prod_k \kappa_k) = (-1)^{(b_1 + \cdots + b_{N-1})}$. The desired result now follows from Equation 7 and the fact that $b_k \equiv b_k^2 \pmod 2$. ☐

Lemma 4.3. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$. Let $\mathcal{P}$ be the set of plugs for $\mathcal{D}$. Let $\tilde{A} \in \mathbb{Z}^{\mathcal{P} \times \mathcal{P}}$ be the matrix defined in Equations 6 and 5. Then $\tilde{A}$ is real symmetric.*

Proof. Let $p_0, p_1 \in \mathcal{P}$ be disjoint plugs; let $\mathbf{t} \in \mathcal{T}(\mathcal{D}_{p_0,p_1})$ be a tiling. We prove that $\mathrm{tw}_{p_0,p_1}(\mathbf{t}) = \mathrm{tw}_{p_1,p_0}(\mathbf{t})$. Indeed, the definitions of $\mathrm{tk}(\mathbf{t})$ and of $\mathrm{inv}(\sigma_{\mathbf{t}})$ are unchanged, so it suffices to prove that $\mathrm{inv}_{\mathrm{bl},p_0,p_1} + \mathrm{inv}_{\mathrm{bl},p_1,p_0} = \mathrm{inv}_{\mathrm{wh},p_0,p_1} + \mathrm{inv}_{\mathrm{wh},p_1,p_0}$. Indeed, $\mathrm{inv}_{\mathrm{bl},p_0,p_1} + \mathrm{inv}_{\mathrm{bl},p_1,p_0} = b_0 b_1 + (b_0 + b_1)(b - b_0 - b_1)$ since it counts pairs $\{i_0, i_1\} \subseteq \{1, \ldots, b\}$ with $h_{\mathrm{bl}}(i_0) \neq h_{\mathrm{bl}}(i_1)$. For the same reason, we also have $\mathrm{inv}_{\mathrm{wh},p_0,p_1} + \mathrm{inv}_{\mathrm{wh},p_1,p_0} = b_0 b_1 + (b_0 + b_1)(b - b_0 - b_1)$ and we are done. □

5. Computing the defect Δ

In this section we first give an estimate for $\Delta(\mathcal{R}_N)$ as a function of N. We then give an explicit formula in special cases. In many cases the defect $\Delta(\mathcal{R})$ is easier to compute than the number of tilings $|\mathcal{T}(\mathcal{R})|$: this is related to determinants being easier to compute than permanents.

Lemma 5.1. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$. For every $p \in \mathcal{P}$, if p contains exactly N unit cubes then the cork $\mathcal{R}_{0,N;\mathbf{p}_\circ,p}$ admits a tiling.*

Proof. The proof is by induction on $b = N/2$, the number of black squares in p. The case $b = 0$ is trivial. For $b > 0$, consider a pair of unit cubes v, w in p, v black, w white, such that the distance between v and w (measured in $\mathcal{G}_{\mathcal{D}}$) is minimal. Let $\tilde{p} = p \smallsetminus \{v, w\} \in \mathcal{P}$. By the induction hypothesis there exists a tiling $\tilde{\mathbf{t}}$ of $\mathcal{R}_{0,\tilde{N};\mathbf{p}_\circ,\tilde{p}}$ for $\tilde{N} = N - 2$. On the first $\tilde{N}$ floors $\mathbf{t}$ coincides with $\tilde{\mathbf{t}}$. In order to construct the last two floors, consider a path of minimal length from v to w. By minimality, this path intersects no other unit cubes in p. For unit cubes not in the path the final two floors are filled with a vertical domino. Along the path we use horizontal dominoes, completing the construction. □

Lemma 5.2. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$, where $n \geqslant 4$. Assume that $\mathcal{D}$ contains a box of the form $3 \times 3 \times 1 \times \cdots \times 1$. Then there exists $N_{\min}$ such that for all $p_0, p_1 \in \mathcal{P}$ we have $|(A^N)_{p_0,p_1}| > |(\tilde{A}^N)_{p_0,p_1}|$ for all $N \geqslant N_{\min}$.*

Proof. The vertical tiling $\mathbf{t}_0$ of $\mathcal{R}_2 = \mathcal{D} \times [0,2]$ satisfies $\mathrm{Tw}(\mathbf{t}_0) = 0$. Replace the vertical dominoes in the $3 \times 3 \times 1 \times \cdots \times 1 \times 2$ box by the dominoes shown in Figure 1 to obtain a tiling $\mathbf{t}_1$ of $\mathcal{R}_2 = \mathcal{D} \times [0,2]$ with $\mathrm{Tw}(\mathbf{t}_1) = 1$. Let $b_{\mathcal{D}}$ be the number of black unit cubes in $\mathcal{D}$. Apply Lemma 5.1 to the full plug $\mathbf{p}_\bullet$ to obtain a tiling $\mathbf{t}_\bullet$ of the cork $\mathcal{R}_{0,2b_{\mathcal{D}};\mathbf{p}_\circ,\mathbf{p}_\bullet}$. The tiling $\mathbf{t}_\bullet$ can be considered a tiling of $\mathcal{R}_{2b_{\mathcal{D}}-1}$. There are therefore tilings of $\mathcal{R}_{N_0}$ of either twist for $N_0 \geqslant 2b_{\mathcal{D}} + 1$.

Take $N_{\min} = 6b_{\mathcal{D}} + 1$ and $N \geqslant N_{\min}$. Assume that p_i contains b_i black unit cubes. Apply Lemma 5.1 to p_0 to obtain a tiling of the cork $\mathcal{R}_{0,2b_0;p_0,\mathbf{p}_\circ}$. Apply Lemma 5.1 to p_1 to obtain a tiling of the cork $\mathcal{R}_{N-2b_1,N;\mathbf{p}_\circ,p_1}$. Clearly, $(N - 2b_1) - 2b_0 \geqslant N_0$. From the previous paragraph, there exist tilings of $\mathcal{R}_{2b_0,N-2b_1;\mathbf{p}_\circ,\mathbf{p}_\circ}$ of either twist. Juxtapose the tilings to obtain two tilings $\tilde{\mathbf{t}}_0, \tilde{\mathbf{t}}_1 \in \mathcal{T}(\mathcal{R}_{0,N;p_0,p_1})$ with contributions of opposite signs to $(\tilde{A}^N)_{p_0,p_1}$, completing the proof. □

Lemma 5.3. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$, where $n \geqslant 4$. Assume that $\mathcal{D}$ contains a box of the form $3 \times 3 \times 1 \times \cdots \times 1$. Then there exist $\lambda > 1$ and $C > 0$ such that*

$$\lim_{N\to\infty} \frac{|\mathcal{T}(\mathcal{R}_N)| - C\lambda^N}{\lambda^N} = 0.$$

Furthermore, $\Delta(\mathcal{R}_N)$ as a function of N is either eventually constant, eventually periodic with period 2 or there exist $\tilde{\lambda} \in (1, \lambda)$ and $\tilde{C} > 0$ such that

$$\limsup_{N\to\infty} \frac{|\Delta(N)|}{\tilde{\lambda}^N} = \tilde{C}.$$

Proof. We apply the Perron–Frobenius Theorem to the matrix A. It follows from Lemma 5.2 that there exists a positive eigenvalue λ such that all other eigenvalues have strictly smaller absolute value. The associated eigenvector has positive entries and therefore $|\mathcal{T}(\mathcal{R}_N)| = (A^N)_{\mathbf{p}_\circ,\mathbf{p}_\circ}$ has a leading term $C\lambda^N$: any other term is exponentially smaller. Since A is symmetric and all entries of A are integers, all eigenvalues are real algebraic integers. If an eigenvalue belongs to $\mathbb{R} \smallsetminus \{-1,0,1\}$, one of its conjugates must have absolute value larger than 1 and therefore $\lambda > 1$. If all eigenvalues belong to $\{-1,0,1\}$, $|\mathcal{T}(\mathcal{R}_N)|$ is eventually periodic with period 1 or 2 and therefore bounded, contradicting the construction in the proof of Lemma 5.2.

Again from Perron–Frobenius, all eigenvalues of $\tilde{A}$ have absolute value smaller than λ. Let $\tilde{\lambda}$ be the maximum absolute value of eigenvalues of $\tilde{A}$. If $\tilde{\lambda} \leqslant 1$ then all eigenvalues belong to $\{-1,0,1\}$ and therefore $\Delta(\mathcal{R}_N)$ is eventually periodic with period 1 or 2. If $\tilde{\lambda} > 1$ then at least one of $\pm\tilde{\lambda}$ is an eigenvalue, implying the last estimate in the statement. □

Corollary 5.4. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$, where $n \geqslant 4$. Assume that $\mathcal{D}$ contains a box of the form $3 \times 3 \times 1 \times \cdots \times 1$. Then*

$$\lim_{N\to\infty} \frac{|\{\mathbf{t} \in \mathcal{T}(\mathcal{R}_N) \mid \operatorname{Tw}(\mathbf{t}) = 0\}|}{|\mathcal{T}(\mathcal{R}_N)|} = \frac{1}{2}, \quad \lim_{N\to\infty} \frac{|\{\mathbf{t} \in \mathcal{T}(\mathcal{R}_N) \mid \operatorname{Tw}(\mathbf{t}) = 1\}|}{|\mathcal{T}(\mathcal{R}_N)|} = \frac{1}{2}.$$

Proof. From Lemma 5.3 we have

$$\lim_{N\to\infty} \frac{\Delta(\mathcal{R}_N)}{|\mathcal{T}(\mathcal{R}_N)|} = 0.$$

The result follows from Equation 3. □

The following result, in a similar spirit, will be needed to prove Corollary 1.4.

Lemma 5.5. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$, where $n \geqslant 4$. Assume that $\mathcal{D}$ contains a $3 \times 3 \times 1 \times \cdots \times 1$ box. There exists $c < 1$ with the following properties. Let M be a fixed positive integer. Let C_N be the number of tilings $\mathbf{t}$ of $\mathcal{R}_N$ with fewer than M vertical floors. Then*

$$\lim_{N\to\infty} \frac{C_N}{c^N |\mathcal{T}(\mathcal{R}_N)|} = 0.$$

Proof. Let A be the adjacency matrix, as above. Let $A_\sharp$ be the corresponding matrix, but not counting vertical floors. We have $|(A_\sharp)_{i,j}| \leqslant A_{i,j}$ for all $i,j \in \mathcal{P}$, with strict inequality for some entries.

Let $\lambda > 0$ be the eigenvalue of A of largest absolute value, as above. There exists $c < 1$ such that all eigenvalues of $A_\sharp$ have absolute value strictly smaller than $c\lambda$. This is our desired c.

Consider an auxiliary c_-, $c_- < c$, such that all eigenvalues of $A_\sharp$ also have absolute value smaller that $c_-\lambda$. Thus, for any $i,j \in \mathcal{P}$,

$$\lim_{N\to\infty} \frac{(A_\sharp^N)_{i,j}}{c_-^N \lambda^N} = 0.$$

Thus, there exists a constant $C_\sharp$ such that $|(A_\sharp^N)_{i,j}| < C_\sharp c_-^N \lambda^N$ for all $i,j \in \mathcal{P}$ and all N.

We need an estimate for C_N. The number of M-tuples $0 < k_1 < \cdots < k_M < N$ is bounded by N^M. For each such M-tuple $(k_1,\ldots,k_M)$, we count the number of tilings of $\mathcal{R}_N$ where vertical floors are allowed only in the positions k_i. We first choose floors and plugs in the positions k_i and k_{i+1}: there are a fixed number K of such choices. We then choose the tiling in each interval: the initial and final plugs p_i and p_{i+1} are now fixed. There are $(A_\sharp^{k_{i+1}-k_i})_{p_i,p_{i+1}} < C_\sharp c_-^{(k_{i+1}-k_i)} \lambda^{(k_{i+1}-k_i)}$ such tilings. Thus, $C_N < N^M C_\sharp c_-^N \lambda^N$; for large N, $C_N \ll c^N \lambda^N$, as desired. □

6. The domino complex $\mathcal{C}_\mathcal{D}$

In this section we complete the construction of the 2-complex $\mathcal{C}_\mathcal{D}$. The construction is very similar to the one performed in [12], thus we skip some details. Recall from Section 4 that tilings of $\mathcal{R}_N = \mathcal{D} \times [0, N]$ correspond to walks of length N in $\mathcal{C}_\mathcal{D}$ from $\mathbf{p}_\circ \in \mathcal{P}$ to $\mathbf{p}_\circ$. We shall now see that $\mathbf{t}_0 \sim \mathbf{t}_1$ if and only if the corresponding continuous paths are homotopic with fixed endpoints.

Consider the graph $\mathcal{C}_{1,\mathcal{D}}$ as a 1-complex. To each self loop (always from $\mathbf{p}_\circ$ to itself) attach the boundary of a Möbius band. Otherwise, we attach boundaries of 2-cells (disks). The 2 cells correspond to flips as described below:

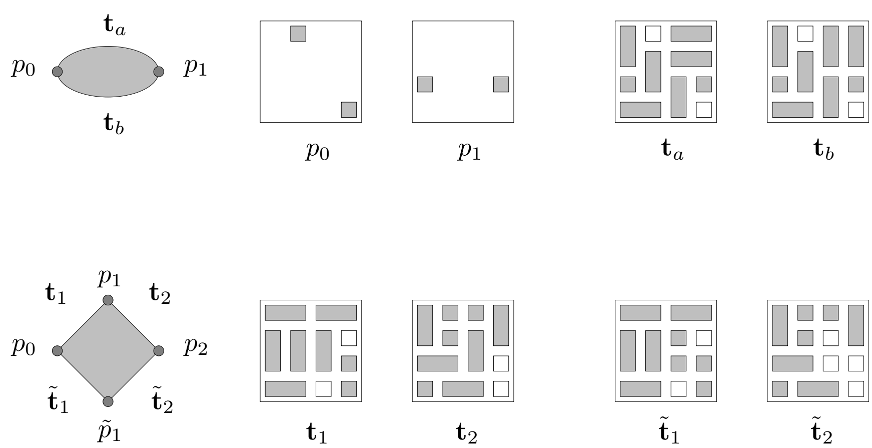

Figure 5. A flip manifests itself in the complex $\mathcal{D}$ as a 2-cell. The figure shows a horizontal and a vertical flip.

First consider horizontal flips. These join two tilings $\mathbf{t}_0, \mathbf{t}_1$ of $\mathcal{D}_{p_0,p_1}$ (where $p_0, p_1 \in \mathcal{P}$ are disjoint plugs). In the complex, p_0 and p_1 are vertices and $\mathbf{t}_0$ and $\mathbf{t}_1$ are 1-cells joining them (in other words, $\mathbf{t}_0$ and $\mathbf{t}_1$ are floors). Attach to the complex a 2-cell whose oriented boundary is $\mathbf{t}_0$ (from p_0 to p_1) followed by $\mathbf{t}_1$ (from p_1 to p_0). Combinatorially, the 2-cell is a bigon. Figure 5 shows an example of such a 2-cell.

Next consider vertical flips. There are now two floors in play. In one tiling, we have $p_0, \mathbf{t}_0, p_1, \mathbf{t}_1, p_2$, where $\mathbf{t}_0 \in \mathcal{T}(\mathcal{D}_{p_0,p_1})$ and $\mathbf{t}_1 \in \mathcal{T}(\mathcal{D}_{p_1,p_2})$. In the other we have $p_0, \tilde{\mathbf{t}}_0, \tilde{p}_1, \tilde{\mathbf{t}}_1, p_2$, where $\tilde{\mathbf{t}}_0 \in \mathcal{T}(\mathcal{D}_{p_0,\tilde{p}_1})$ and $\tilde{\mathbf{t}}_1 \in \mathcal{T}(\mathcal{D}_{\tilde{p}_1,p_2})$. The plug $\tilde{p}_1$ is obtained from p_1 by removing two adjacent unit cubes. These two unit cubes form dominoes in both $\tilde{\mathbf{t}}_0$ and $\tilde{\mathbf{t}}_1$. Again, attach to the complex a 2-cell whose oriented boundary is $\mathbf{t}_0$ (from p_0 to p_1), $\mathbf{t}_1$ (from p_1 to p_2), $\tilde{\mathbf{t}}_1$ (from p_2 to $\tilde{p}_1$) and $\tilde{\mathbf{t}}_0$ (from $\tilde{p}_1$ to p_0). Combinatorially, the 2-cell is a square. Figure 5 also shows an example of this other kind of 2-cell.

By construction, if two tilings $\mathbf{t}_0, \mathbf{t}_1$ of $\mathcal{R}_{0,N;p_0,p_N}$ differ by a flip, the two corresponding (continuous) paths are homotopic. Indeed, we added a 2-cell which guarantees just that. Also, if a tiling $\mathbf{t}_1$ of $\mathcal{R}_{0,N+2;p_0,p_N}$ is obtained from a tiling $\mathbf{t}_0$ of $\mathcal{R}_{0,N;p_0,p_N}$ by inserting two vertical floors (at any position), the two paths are trivially homotopic. The converse statement is similar. Thus, $G_\mathcal{D}$ is naturally identified with the fundamental group $\pi_1(\mathcal{C}_\mathcal{D}, \mathbf{p}_\circ)$.

There is a natural surjective map $G_\mathcal{D} \to \mathbb{Z}/(2)$ taking a tiling of $\mathcal{R}_{0,N;p_0,p_N}$ to N mod 2. The kernel of this map is $G^+_\mathcal{D} < G_\mathcal{D}$, a normal subgroup of index 2.

Since the complex is finite, the group $G_{\mathcal{D}}$ is finitely presented. The immediate construction is far too complicated, however. Later we shall significantly improve this situation.

7. Hamiltonian regions and generators of $G_{\mathcal{D}}$

The results from this section will be used repeatedly to prove regularity of regions, or, more generally, to compute the domino group. We recall the following fact.

Fact 7.1. *Let $\mathcal{R} \subset \mathbb{R}^2$ be a planar balanced quadriculated region. Let $\mathbf{t}_0, \mathbf{t}_1$ be tilings of $\mathcal{R}$. Then $\mathbf{t}_0 \approx \mathbf{t}_1$ if and only if* $\operatorname{Flux}(\mathbf{t}_0) = \operatorname{Flux}(\mathbf{t}_1)$.

For the proof of Fact 7.1, see [14, 16]. The general concept of flux will not be required; we will clarify the meaning in special cases when it comes up.

A cubiculated region $\mathcal{D} \subset \mathbb{R}^{n-1}$ is *Hamiltonian* if the graph $\mathcal{G}_{\mathcal{D}}$ admits a Hamiltonian path. A fixed Hamiltonian path $\gamma_0 = (s_1, \ldots, s_M)$ is usually assumed; here $M = |\mathcal{D}|$ and the s_i are unit cubes.

Example 7.2. Any box $\mathcal{D} = [0, L_1] \times \cdots \times [0, L_{n-1}]$ is Hamiltonian. We construct an explicit path recursively on n. For $n = 2$, the path in $[0, L_1]$ is given by $s_i = [i-1, i]$. Assume a path $\gamma_0 = (s_1, \ldots, s_M)$ given in $[0, L_1] \times \cdots \times [0, L_{n-1}]$, where $M = L_1 \cdots L_{n-1}$. We construct a path $\tilde{\gamma}_0$ in $[0, L_1] \times \cdots \times [0, L_n]$. The number of unit cubes in the new box is $\tilde{M} = M L_n$. For $\tilde{k} \in \mathbb{Z}$, $1 \leqslant \tilde{k} \leqslant \tilde{M}$, let $x_n = \lceil \tilde{k}/M \rceil$ and $k = \tilde{k} - (x_n - 1)M$ if x_n is odd and $k = 1 + x_n M - \tilde{k}$ if x_n is even. Define $\tilde{s}_{\tilde{k}} = s_k \times [x_n - 1, x_n]$. The next example is a special case.

Example 7.3. Some small examples deserve special attention, particularly $\mathcal{D} = [0,2]^2 \times [0, L]$, $L \geqslant 2$. The construction from Example 7.2 applies, but a variation is easier to draw.

Consider the quadriculated cylinder $\hat{\mathcal{D}} = (\mathbb{R}/(4\mathbb{Z})) \times [0, L]$: the bipartite graphs $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{G}_{\hat{\mathcal{D}}}$ are isomorphic. It follows that the bipartite graphs $\mathcal{G}_{\mathcal{R}_N}$ and $\mathcal{G}_{\hat{\mathcal{R}}_N}$ are also isomorphic (for any $N \in \mathbb{N}^*$), where $\hat{\mathcal{R}}_N = \hat{\mathcal{D}} \times [0, N]$ is a 3D cubiculated manifold.

Tilings of $\hat{\mathcal{R}}_N$ can be represented as in Figure 6. Here, floors are shown sequentially. The quadriculated cylinder $\hat{\mathcal{D}}$ is represented by a rectangle where the right and left sides are identified (as in a Mercator map). Each floor f_i is of the form $f_i = (p_{i-1}, f_i^*, p_i)$. Here, as in Equation 4, $p_{i-1}, p_i \in \mathcal{P}_{\hat{\mathcal{D}}}$ are disjoint plugs and f_i^* is a tiling of $\hat{\mathcal{D}}_{p_{i-1}, p_1}$.

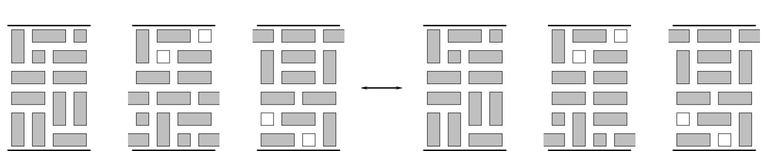

Figure 6. Two tilings of $\hat{\mathcal{R}}_N$ for $N = 3$ and $L = 6$. The two tilings differ by a pseudoflip in the fourth row of the second floor.

There exists an important difference between $\mathcal{R}_N$ and $\hat{\mathcal{R}}_N$, however. Some flips in $\mathcal{R}_N$ are represented in $\hat{\mathcal{R}}_N$ not in the usual way, but as *pseudoflips*. In a pseudoflip, a row (of length 4) of $\hat{\mathcal{D}}$ is rotated by one unit, as shown in Figure 6. In $\mathcal{R}_N$, which has higher dimension, a pseudoflip is an honest flip.

The regions $\mathcal{D} = [0,2]^2 \times [0, L]$ and $\hat{\mathcal{D}} = C_4 \times [0, L]$ are Hamiltonian. Figure 7 shows Hamiltonian paths in $\tilde{\mathcal{D}}$ for $L = 3, 4, 5$.

Recall that a domino is *horizontal* if it is contained in a single floor and is *vertical* otherwise. We say that a horizontal domino *respects the path* if and only if it corresponds to an edge along the path; vertical dominoes always respect the path. A tiling *respects the path* if and only if it consists only of dominoes which respect the path.

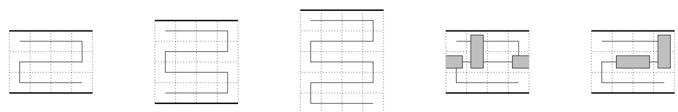

FIGURE 7. The first three diagrams show Hamiltonian paths γ_0 in the quadriculated surfaces $\hat{\mathcal{D}} = C_4 \times [0, L]$ for $L = 3, 4, 5$. The fourth diagram shows two horizontal dominoes which do not respect the path. The fifth diagram shows two horizontal dominoes which respect the path.

Consider a domino $d \subset \mathcal{D}$ which does not respect the path. We have $d = s_{i_{d,-}} \cup s_{i_{d,+}} \subset \mathcal{D}$ with $i_{d,-} + 1 < i_{d,+}$. The domino d decomposes the path γ_0 into intervals (finite sets of integers):

$$I_{d;-} = \mathbb{Z} \cap [1, i_{d,-} - 1], \quad I_{d;+} = \mathbb{Z} \cap [i_{d,+} + 1, M],$$
$$I_{d;0} = \mathbb{Z} \cap [i_{d,-} + 1, i_{d,+} - 1].$$

The set $I_{d;0}$ always has even and positive cardinality. A plug $p \in \mathcal{P}$ is *compatible* with d if and only if it does not include a square of d. If p is a plug compatible with d define:

$$\operatorname{flux}(d;p) = (\operatorname{flux}_-(d;p), \operatorname{flux}_0(d;p), \operatorname{flux}_+(d;p)), \quad \operatorname{flux}_j(d;p) = \sum_{i \in I_{d;j}, s_i \subset p} (-1)^i.$$

Let $H \subset \mathbb{Z}^3$ be the perpendicular lattice to $(1, 1, 1)$; let $\Phi_d \subset H$ be the finite set of values of $\operatorname{flux}(d;p)$ for $p \in \mathcal{P}$ compatible with d.

LEMMA 7.4. *Consider a Hamiltonian region $\mathcal{D} \subset \mathbb{R}^{n-1}$ with a fixed path γ_0.*

(1) *Let $d \subset \mathcal{D}$ be a domino which does not respect the path. Let $p \in \mathcal{P}$ be a plug compatible with d. Then, for sufficiently large even N there exists a tiling $\mathbf{t}_{d;p}$ of $\mathcal{R}_{2N}$ with the following properties. There exists a unique domino in $\mathbf{t}_{d;p}$ which does not respect the path: $d \times [N-1, N]$. The plug of $\mathbf{t}_{d;p}$ at height $N-1$ is p.*
(2) *Let $d \subset \mathcal{D}$ be a domino which does not respect the path. Let $p_0, p_1 \in \mathcal{P}$ be plugs compatible with d. Let $\mathbf{t}_{d;p_0}, \mathbf{t}_{d;p_1}$ be tilings of $\mathcal{R}_{2N}$ satisfying the conditions of the first item. If $\operatorname{flux}(d;p_0) = \operatorname{flux}(d;p_1)$ then $\mathbf{t}_{d;p_0} \approx \mathbf{t}_{d;p_1}$. Also, the sequence of flips from $\mathbf{t}_{d;p_0}$ to $\mathbf{t}_{d;p_1}$ can be chosen so as to keep the domino $d \times [N-1, N]$ fixed and all other dominoes respect the path.*

Proof. A tiling of $\mathcal{R}_{2N}$ which respects the path can be *unfolded* to obtain a tiling of $\tilde{\mathcal{R}}_{2N} = [0, M] \times [0, 2N]$. A horizontal domino in $\mathcal{R}_{2N}$ of the form $\tilde{d} \times [j-1, j]$, $\tilde{d} = s_i \cup s_{i+1} \subset \mathcal{D}$, is taken to $[i-1, i+1] \times [j-1, j] \subset \tilde{\mathcal{R}}_{2N}$. A vertical domino in $\mathcal{R}_{2N}$ of the form $s_i \times [j-1, j+1]$ is taken to $[i-1, i] \times [j-1, j+1] \subset \tilde{\mathcal{R}}_N$.

Similarly, consider a tiling $\mathbf{t}$ of $\mathcal{R}_{2N}$ such that there exists a unique domino in $\mathbf{t}_{d;p}$ which does not respect the path: $d \times [N-1, N]$. The tiling $\mathbf{t}$ can be unfolded to obtain a tiling $\tilde{\mathbf{t}}$ of the planar region $\tilde{\mathcal{R}}_{2N,d}$:

$$\tilde{\mathcal{R}}_{2N,d} = ([0, M] \times [0, 2N]) \smallsetminus (s_- \cup s_+) \subset \mathbb{R}^2,$$
$$s_- = [i_{d,-} - 1, i_{d,-}] \times [N-1, N], \qquad s_+ = [i_{d,+} - 1, i_{d,+}] \times [N-1, N].$$

Conversely, a tiling $\tilde{\mathbf{t}}$ of $\tilde{\mathcal{R}}_{2N,d}$ can be folded to obtain a tiling $\mathbf{t}$ of $\mathcal{R}_{2N}$ with the properties above.

For the first item, the information about plugs reduces the problem to tiling two similar contractible planar regions. The first region is obtained from the rectangle $[0, M] \times [0, N-1]$ by removing from row $[0, M] \times [N-2, N-1]$ the unit squares contained in p. The second region is obtained from the rectangle $[0, M] \times [N-1, 2N]$ by removing from row $[0, M] \times [N-1, N]$ both the unit squares contained in p and the domino d. This is discussed in [12]; see also [16].

For the second item, unfold the tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ to obtain tilings $\tilde{\mathbf{t}}_0$ and $\tilde{\mathbf{t}}_1$ of the planar region $\tilde{\mathcal{R}}_{2N,d}$. The condition $\operatorname{flux}(d; p_0) = \operatorname{flux}(d; p_1)$ is translated to $\operatorname{Flux}(\tilde{\mathbf{t}}_0) = \operatorname{Flux}(\tilde{\mathbf{t}}_1)$. From Fact 7.1, $\tilde{\mathbf{t}}_0 \approx \tilde{\mathbf{t}}_1$. Take the sequence of flips for the planar problem and fold back to obtain the desired sequence of flips in $\mathcal{R}_{2N}$. □

Consider a domino $d \subset \mathcal{D}$ not respecting the path and $\phi \in \Phi_d \subset H \subset \mathbb{Z}^3$. Choose $p \in \mathcal{P}$, p compatible with d, $\operatorname{flux}(d; p) = \phi$. Apply the first item of Lemma 7.4 to obtain a tiling $\mathbf{t}_{d;\phi} = \mathbf{t}_{d;p}$ with the properties listed in that item. Notice that the second item implies that, for fixed N (but independently of p), all such tilings are mutually connected by sequences of flips.

Lemma 7.5. *The family of tilings* $(\mathbf{t}_{d;\phi})$, $d \subset \mathcal{D}$ *not respecting the path* γ_0, $\phi \in \Phi_d$, *generates the subgroup* $G^+_{\mathcal{D}} < G_{\mathcal{D}}$.

Proof. Recall that $G^+_{\mathcal{D}} < G_{\mathcal{D}}$ is a normal subgroup of index 2, the kernel of the natural surjective map $G_{\mathcal{D}} \to \mathbb{Z}/(2)$ (parity of length of walk). The proof follows with very slight adaptations the proof of Corollary 8.6 in [12]. □

8. Irregularity of $\mathcal{D} = [0, 2]^3$

We now discuss the smallest non trivial example: see Example 2.2.

Lemma 8.1. *Let* $\mathcal{D} = [0, 2]^3$. *There exists a surjective map* $\operatorname{Tw}_{\mathbb{Z}} : G_{\mathcal{D}} \to \mathbb{Z}$ *such that* $\operatorname{Tw}(\mathbf{t}) = \operatorname{Tw}_{\mathbb{Z}}(\mathbf{t}) \bmod 2$ *for any tiling* $\mathbf{t}$ *of* $\mathcal{D} \times [0, N]$, $N \in \mathbb{N}^*$. *In particular,* $\mathcal{D}$ *is not regular.*

Proof. Consider a domino d and a square s contained in $\hat{\mathcal{D}} = C_4 \times [0, 2]$: we define $\tau(d, s) \in \{-\frac{1}{4}, 0, \frac{1}{4}\}$ as in Figure 8. For other configurations, $\tau(d, s) = 0$. Thus, $\tau(d, s) \neq 0$ if and only if d and s are disjoint, $d \subset \hat{\mathcal{D}}$ is in the C_4 direction and a projection onto C_4 takes s to a subset of d.

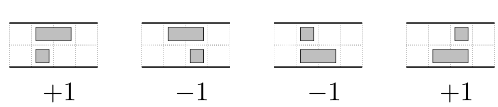

Figure 8. The value of $4\tau(d, s)$ in four examples. The sign depends on two bits: the horizontal position of the square and the relative position of the square and domino. We can give signs in a consistent way only in this small case.

Recall that a plug $p \in \mathcal{P}_{\hat{\mathcal{D}}}$ is a balanced subset of $\hat{\mathcal{D}}$. If $p, \tilde{p} \in \mathcal{P}_{\hat{\mathcal{D}}}$ are disjoint plugs then $\hat{\mathcal{D}}_{p,\tilde{p}} = \hat{\mathcal{D}} \smallsetminus (p \cup \tilde{p})$. For disjoint plugs $p, \tilde{p} \in \mathcal{P}_{\hat{\mathcal{D}}}$ and $f \in \mathcal{T}(\hat{\mathcal{D}}_{p,\tilde{p}})$ define

$$\tau(f, p) = \sum_{d \in f, s \in p} \tau(d, s) \in \frac{1}{4}\mathbb{Z}; \qquad \tau(f; p, \tilde{p}) = \tau(f, \tilde{p}) - \tau(f, p) \in \frac{1}{4}\mathbb{Z}. \tag{8}$$

Draw a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_N)$ as a sequence of floors, as in Figure 6. A tiling is therefore an alternating sequence of plugs and floors,

$$\mathbf{t} = (p_0, \ldots, f_i, p_i, f_{i+1}, p_{i+1}, \ldots, p_N),$$

with $p_i \in \mathcal{P}_{\hat{\mathcal{D}}}$ (for all i), $p_0 = p_N = \mathbf{p}_\circ$ and $f_i \in \mathcal{T}(\hat{\mathcal{D}}_{p_{i-1},p_i})$. Define

$$\mathrm{Tw}_{\mathbb{Z}}(\mathbf{t}) = \sum_{0<j\leqslant N} \tau^u(\mathrm{floor}_j(\mathbf{t}); \mathrm{plug}_{j-1}(\mathbf{t}), \mathrm{plug}_j(\mathbf{t})).$$

It is now not hard to verify that $\mathrm{Tw}_{\mathbb{Z}}(\mathbf{t}) \in \mathbb{Z}$ for any tiling $\mathbf{t}$ and that $\mathrm{Tw}_{\mathbb{Z}}(\mathbf{t})$ is invariant under flips and pseudoflips. □

Remark 8.2. Recall from Example 2.2 that the box $[0,2]^4$ admits 272 tilings, among them 8 which are isolated, i.e. admit no flip. Figure 9 shows an example; the others are obtained by rotation and reflection. See also the second tiling in Figure 3. With the concept of twist as $\mathrm{Tw}_{\mathbb{Z}}$, defined in Lemma 8.1, four of the 8 isolated tilings have twist $+1$ and four have twist -1.

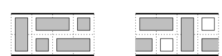

Figure 9. An isolated tiling of the box $[0,2]^4$, represented here as $\hat{\mathcal{R}}_2$.

Let $\mathbf{t}_i$, $i \in \{1,2,3,4\}$, be the four isolated tilings with twist $+1$. The tilings with twist -1 are the reflections $\mathbf{t}_i^{-1}$. Let $\mathbf{t}_{\mathrm{thin}}$ be a tiling of $\mathcal{R}_1$ (they are all $\approx$-equivalent). Let $\mathbf{t}_{\mathrm{vert}}$ be the vertical tiling of $\mathcal{R}_6$. The brute force study of tilings of $\mathcal{R}_6$ shows that, for all $i \in \{1,2,3,4\}$,

$$\mathbf{t}_1 * \mathbf{t}_{\mathrm{thin}} * \mathbf{t}_{\mathrm{thin}} * \mathbf{t}_i^{-1} \approx \mathbf{t}_1 * \mathbf{t}_{\mathrm{thin}} * \mathbf{t}_i^{-1} * \mathbf{t}_{\mathrm{thin}} \approx \mathbf{t}_{\mathrm{vert}}.$$

This implies that the four tilings $\mathbf{t}_i$ represent the same element of the domino group $G_{\mathcal{D}}$. Also, $\mathbf{t}_1 * \mathbf{t}_{\mathrm{thin}} * \mathbf{t}_{\mathrm{thin}} * \mathbf{t}_i$ has twist 2 and belongs to a component of size 98144.

Lemma 8.3. *Let $\mathcal{D} = [0,2]^3$. The domino group $G_{\mathcal{D}}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/(2)$. Any tiling of $[0,2]^4$ which admits no flips (such as the one in Figure 9) is a generator of the $\mathbb{Z}$ component. Any tiling of $[0,2]^3 \times [0,1]$ is a generator of the $\mathbb{Z}/(2)$ component.*

Proof. Consider the homomorphism from $G_{\mathcal{D}}$ to $\mathbb{Z} \oplus \mathbb{Z}/(2)$ taking $\mathbf{t} \in \mathcal{T}(\mathcal{R}_N)$ to $(\mathrm{Tw}_{\mathbb{Z}}(\mathbf{t}), N \bmod 2)$: this homomorphism is clearly surjective.

Let $\mathbf{t}_1$ and $\mathbf{t}_{\mathrm{thin}}$ be as in Remark 8.2. Consider the homomorphism from $\mathbb{Z} \oplus \mathbb{Z}/(2)$ to $G_{\mathcal{D}}$ taking (u,v) to $\mathbf{t}_1^u * \mathbf{t}_{\mathrm{thin}}^v$. This homomorphism is clearly injective, all we have to do is show that it is surjective.

Consider the generators $\mathbf{t}_{d;\phi}$ of $G_{\mathcal{D}}^+$ constructed in Lemma 7.5. Computations show that each $\mathbf{t}_{d;\phi}$ is homotopic to a power of $\mathbf{t}_1$; i.e. $\mathbf{t}_1 * \mathbf{t}_1 * \cdots * \mathbf{t}_1$. □

9. Regularity of $\mathcal{D} = [0,2]^2 \times [0,L]$, $L \geqslant 3$

We now discuss other small examples, the boxes $\mathcal{D} = [0,2]^2 \times [0,L]$ for $L \geqslant 3$, as in Example 2.3.

Lemma 9.1. *The box $\mathcal{D} = [0,2]^2 \times [0,L]$ is regular for $L \geqslant 3$.*

We shall need the following facts. Fact 9.2 is a special case of the first main theorem in [12].

Fact 9.2. *The rectangle $\mathcal{D}_0 = [0,4] \times [0,L]$ is regular for $L \geqslant 3$.*

Fact 9.3. *Let $\mathcal{D} = [0,2]^2 \times [0,3]$ and $\mathcal{R}_3 = \mathcal{D} \times [0,3]$. If $\mathbf{t}_0$ and $\mathbf{t}_1$ are tilings of $\mathcal{R}_3$ with $\mathrm{Tw}(\mathbf{t}_0) = \mathrm{Tw}(\mathbf{t}_1) = 1$ then $\mathbf{t}_0 \approx \mathbf{t}_1$.*

Fact 9.3 can be verified by brute force. As mentioned in Example 2.3, all tilings of $\mathcal{R}_3$ of twist 1 form a connected component of size 99280.

Proof of Lemma 9.1. We follow the construction in the previous section, particularly Lemma 7.5. We use the Hamiltonian paths in Example 7.3. Let $\mathbf{t}_0 = \mathbf{t}_{\mathrm{vert}}$ be the vertical tiling of $\mathcal{R}_2$. Let $\mathbf{t}_{\pm 1}$ be the tilings of $\mathcal{R}_4$ shown in Figure 10 (for $L > 3$, the other rows are similar to the third row in the figure). These two tilings are of the form $\mathbf{t}_{+1} = \mathbf{t}_{d;\phi}$ and $\mathbf{t}_{-1} = \mathbf{t}_{d;\phi'}$ where the domino d is the only one which does not respect the path. We have $\mathrm{tw}(\mathbf{t}_{+1}) = \mathrm{tw}(\mathbf{t}_{-1}) = 1 \in \mathbb{Z}/(2)$. It follows from Fact 9.3 (indeed, from Remark 2.1) that $\mathbf{t}_{+1} \approx \mathbf{t}_{-1}$. We prove that $\mathbf{t}_{+1}$ generates $G^+_{\mathcal{D}} \approx \mathbb{Z}/(2)$, which implies regularity.

Figure 10. Two tilings $\mathbf{t}_{+1}$ and $\mathbf{t}_{-1}$ of $\mathcal{D} \times [0,4]$ for $\mathcal{D} = [0,2]^2 \times [0,3]$.

Notice that, for given L, this already reduces the proof to a finite and reasonably small computation. Indeed, for each domino d not respecting the path compute $\Phi_d \subset H$. For each pair (d,ϕ), $\phi \in \Phi_d$, construct a tiling $\mathbf{t}_{d;\phi}$. Compute the twist of these tilings. For each pair (d,ϕ), we must verify that if $\mathrm{tw}(\mathbf{t}_{d,\phi}) = s$ then $\mathbf{t}_{d,\phi} \sim \mathbf{t}_s$.

We now address the general case $L \geqslant 3$. If the domino d does not cross the sides of the rectangle, the other dominoes will also not cross (they respect the path). We may therefore consider $\mathbf{t}_{d;\phi}$ to be a tiling of $[0,4] \times [0,L] \times [0,N]$. With this interpretation, we are fully in the scenario of [12], and we know that $[0,4] \times [0,L]$ is regular: this is Fact 9.2. We stress that *regularity* in the previous sentence means *regularity in the 3d sense*. In other words, let $\mathcal{D}_0$ be the quadriculated disk $\mathcal{D}_0 = [0,4] \times [0,L]$; let $G_{\mathcal{D}_0}$ be the domino group of $\mathcal{D}_0$ (as defined in [12]): we have $G_{\mathcal{D}_0} \approx \mathbb{Z} \oplus \mathbb{Z}/(2)$. Thus, regularity of $\mathcal{D}_0$ implies that $\mathbf{t}_{d;\phi}$ is homotopic (still with basis $\mathcal{D}_0 = [0,4] \times [0,L]$) to a product of copies of $\mathbf{t}_{+1}$ and $\mathbf{t}_{-1}$. This in turn implies (now with basis $[0,2]^2 \times [0,L]$) that $\mathbf{t}_{d,\phi}$ is homotopic to a product of copies of $\mathbf{t}_{+1}$ and $\mathbf{t}_{-1}$. With basis $[0,2]^2 \times [0,L]$, $\mathbf{t}_{+1}$ and $\mathbf{t}_{-1}$ are homotopic to each other and both have degree 2 (from Fact 9.3). Thus, $\mathbf{t}_{d,\phi}$ is homotopic to either $\mathbf{t}_1$ or $\mathbf{t}_0$. In other words, $\mathbf{t}_{d,\phi} \sim \mathbf{t}_s$ where $s = \mathrm{tw}(\mathbf{t}_{d,\phi})$. We are then left with checking the L horizontal dominoes which cross the side of the rectangle. Figure 11 shows these dominoes for $L = 3$.

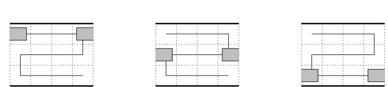

Figure 11. Three dominoes which cross the side of the rectangle.

Notice that for these dominoes, the central interval $I_{d;0}$ has exactly two elements and therefore $|\phi_0| \leqslant 1$, $\phi_0 = \mathrm{flux}_0(d;p)$. We first address the case $\phi_0 = 0$. In this case we may assume that the two elements of $I_{d;0}$ are covered by a domino in $\mathbf{t}_{d;\phi}$ (we are using Fact 7.1 here), as in the first tiling of Figure 12. A pseudoflip then takes $\mathbf{t}_{d;\phi}$ to a tiling $\mathbf{t}$ which everywhere respects the path. We thus have $\mathbf{t}_{d;\phi} \approx \mathbf{t} \sim \mathbf{t}_0$, taking care of this case.

For the case $|\phi_0| = 1$, we may assume without loss of generality that $\phi_0\phi_+ < 0$ (and γ_0 moves from top to bottom). We may therefore assume that we have a configuration similar to the one in Figure 13 (other dominoes respecting the path are not shown); different configurations are minor variations: we show them for $L = 4$ in Figure 14. A sequence of flips (and pseudoflips) takes us to a tiling $\mathbf{t}$ of $[0,4] \times [0,L] \times [0,N]$. As in

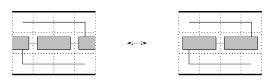

Figure 12. The case $\phi_0 = 0$.

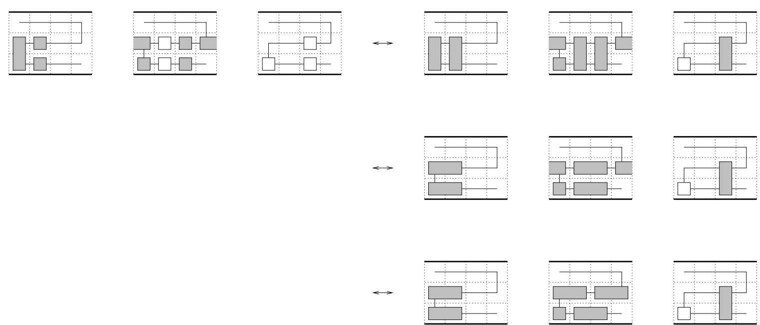

Figure 13. The case $|\phi_0| = 1$.

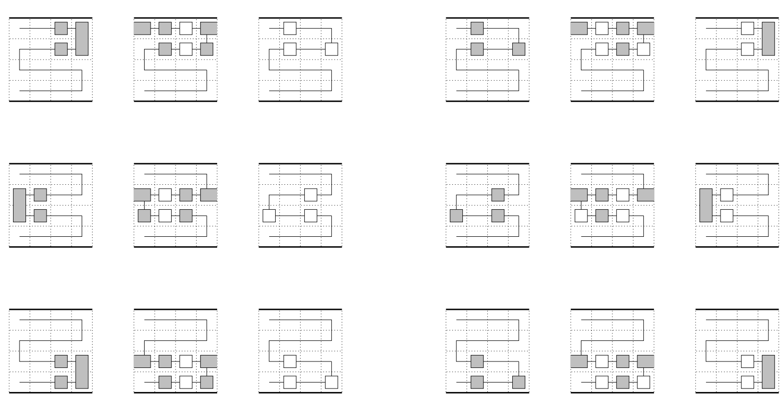

Figure 14. The case $|\phi_0| = 1$, other configurations.

the previous case, $\mathbf{t}$ is equivalent to a product of copies of $\mathbf{t}_{+1}$ and $\mathbf{t}_{-1}$, completing this last case and the proof of the lemma. □

10. Regularity of boxes for $n = 4$

In this section we prove regularity for almost all boxes in $n = 4$. We shall need the following result which appears in [12].

Fact 10.1. *Let $\tilde{\mathcal{D}} = [0, L_a] \times [0, L_b] \subset \mathbb{R}^2$ be a rectangle with $L_a, L_b \in \mathbb{N}^*$, $L_a, L_b \geqslant 3$, $L_a L_b$ even. Then the rectangle $\tilde{\mathcal{D}}$ is regular.*

Thus, if N is even, a tiling $\mathbf{t}$ of $\tilde{\mathcal{D}} \times [0, N]$ is homotopic (i.e. $\sim$-equivalent) to a product of finitely many copies of the tilings $\mathbf{t}_{+1}$ and $\mathbf{t}_{-1}$ of $\tilde{\mathcal{D}} \times [0, 4]$, shown in Figure 15 for $L_a = L_b = 4$. In general, the upper 2×3 rectangle is as shown and the rest is filled with dominoes respecting the path.

Consider two quadriculated or cubiculated regions $\mathcal{R}_1$, $\mathcal{R}_2$ with Hamiltonian paths γ_1 and γ_2. We say that $\mathcal{R}_2$ is obtained by *folding* $\mathcal{R}_1$ if and only if $|\mathcal{R}_1| = |\mathcal{R}_2|$ and, for all k_0, k_1, if $\gamma_1(k_0)$ and $\gamma_1(k_1)$ are adjacent then $\gamma_2(k_0)$ and $\gamma_2(k_1)$ are also adjacent. We also say that $\mathcal{R}_1$ is obtained from $\mathcal{R}_2$ by *unfolding*. A trivial example is that a

Figure 15. The tilings $\mathbf{t}_{\pm 1}$ for $L_a = L_b = 4$.

Hamiltonian region is obtained by folding the path itself. The following example will be used more than once.

Example 10.2. The box $\mathcal{R}_2$ below (of dimension n) is obtained by folding $\mathcal{R}_1$ (of dimension $n-1$):

$$\mathcal{R}_1 = [0, L_1] \times \cdots \times [0, L_k L_{k+1}] \times \cdots \times [0, L_n],$$
$$\mathcal{R}_2 = [0, L_1] \times \cdots \times [0, L_k] \times [0, L_{k+1}] \times \cdots \times [0, L_n],$$

The Hamiltonian paths γ_i are defined as in Example 7.2.

If $\mathcal{R}_2$ is obtained from $\mathcal{R}_1$ by folding, any tiling of $\mathcal{R}_1$ can be folded to define a tiling of $\mathcal{R}_2$. More precisely, if $\gamma_1(k_0)$ and $\gamma_1(k_1)$ form a domino (contained in $\mathcal{R}_1$) then $\gamma_2(k_0)$ and $\gamma_2(k_1)$ also form a domino (contained in $\mathcal{R}_2$). The converse may be true or not: a tiling of $\mathcal{R}_2$ may or may not admit unfolding to $\mathcal{R}_1$.

Theorem 10.3. *Consider a cubiculated box $\mathcal{D} = [0, L_1] \times [0, L_2] \times [0, L_3]$, $L_i \geqslant 2$, $L_1 L_2 L_3$ even. The box $\mathcal{D}$ is regular* except *if $L_1 = L_2 = L_3 = 2$.*

Proof. The cases where at least two of the L_i equal 2 have already been discussed. We may therefore assume $L_1, L_3 \geqslant 3$.

Consider the rectangles $\mathcal{D}_1 = [0, L_1 L_2] \times [0, L_3]$, with Hamiltonian path γ_1, and $\mathcal{D}_2 = [0, L_1] \times [0, L_2 L_3]$, with Hamiltonian path γ_2. As we saw in Example 10.2, there is a *folding* procedure from $\mathcal{D}_i$ to $\mathcal{D}$ and an *unfolding* procedure from $\mathcal{D}$ to $\mathcal{D}_i$ (for $i \in \{1, 2\}$).

A tiling of $\mathcal{D}_i \times [0, N]$ (for $i \in \{1, 2\}$) can always be folded to obtain a tiling of $\mathcal{D} \times [0, N]$. Both $\mathcal{D}_1$ and $\mathcal{D}_2$ are rectangles satisfying the conditions of Fact 10.1. We therefore have tilings $\mathbf{t}_{\pm 1}$ of $\mathcal{D}_i \times [0, 4]$, contructed as in Figure 15 and satisfying $\operatorname{Tw}_{\mathbb{Z}}(\mathbf{t}_{\pm 1}) = \pm 1$ (notice that $\mathcal{D}_i \times [0, 4]$ has dimension 3, so that we are in the situation where twist assumes values in $\mathbb{Z}$). Fold them to obtain tilings $\mathbf{t}_{\pm 1; i}$ of $\mathcal{D} \times [0, 4]$. If the tilings $\mathbf{t}_{\pm 1}$ of $\mathcal{D}_i \times [0, 4]$ are constructed as in Figure 15 then $\mathbf{t}_{\pm 1; i}$ are essentially tilings of the corner $[0, 3] \times [0, 2]^2 \times [0, 3]$ box. More precisely: every domino in $\mathbf{t}_{\pm 1; i}$ is either contained in the box above or disjoint from it; the dominoes outside the box are the same in all four tilings and respect the path. The tilings $\mathbf{t}_{\pm 1; i}$ have twist $1 \in \mathbb{Z}/(2)$. Thus, from Fact 9.3, they are all equivalent (i.e. $\mathbf{t}_{+1;1} \approx \cdots \approx \mathbf{t}_{-1;2}$). Let $\mathbf{t}_1 = \mathbf{t}_{+1;1}$, a tiling of $\mathcal{D} \times [0, 4]$: we have $\mathbf{t}_1 * \mathbf{t}_1 = e$ (in the domino group). We claim that $\mathbf{t}_1$ generates $G^+_{\mathcal{D}}$ (the proof of the claim will complete the proof the theorem).

A domino $d \subset \mathcal{D}$, formed by unit cubes $\gamma_0(k_0)$ and $\gamma_0(k_1)$, *can be unfolded* to $\mathcal{D}_i$ ($i \in \{1, 2\}$) if and only if $\gamma_i(k_0)$ and $\gamma_i(k_1)$ are adjacent. Notice that a domino in the direction e_1 respects the path and can therefore be unfolded to both $\mathcal{D}_1$ and $\mathcal{D}_2$. A domino in the direction e_2 can always be unfolded to $\mathcal{D}_2$ but usually not to $\mathcal{D}_1$. A domino in the direction e_3 can always be unfolded to $\mathcal{D}_1$ but usually not to $\mathcal{D}_2$. In particular, every domino that respects the path can be unfolded to either one among $\mathcal{D}_1$ and $\mathcal{D}_2$ and every domino can be unfolded to at least one among $\mathcal{D}_1$ and $\mathcal{D}_2$.

As in the construction detailed in Section 7, let $d \subset \mathcal{D}$ be a domino that does not respect the path γ_0, let ϕ be a possible value of the flux and let $\mathbf{t}_{d;\phi}$ be the corresponding tiling of $\mathcal{D} \times [0, N]$. The tiling $\mathbf{t}_{d;\phi}$ has a unique domino which does not respect the path and therefore can be unfolded to obtain a tiling $\tilde{\mathbf{t}}$ of $\mathcal{D}_i \times [0, N]$ for some choice of $i \in \{1, 2\}$. From Fact 10.1, there exists a finite sequence of flips taking $\tilde{\mathbf{t}} * \mathbf{t}_{\text{vert}}$ to a product of finitely many copies of $\mathbf{t}_{\pm 1}$ (tilings of $\mathcal{D}_i \times [0, \tilde{N}]$ for some even $\tilde{N} \geqslant N$). Fold this sequence of flips to obtain a similar sequence from $\mathbf{t}_{d;\phi} * \mathbf{t}_{\text{vert}}$ to a product of finitely many copies of $\mathbf{t}_{\pm 1;i}$ (tilings of $\mathcal{D} \times [0, \tilde{N}]$). From what we saw above, $\mathbf{t}_{d;\phi}$ is then equivalent to some power of $\mathbf{t}_1$. This completes the proof of the claim and of the theorem. □

11. Regularity of boxes for $n > 4$

The following lemma completes the proof of Theorem 1.1.

Lemma 11.1. *Consider $n > 4$. Consider a cubiculated box $\mathcal{D} = [0, L_1] \times \cdots \times [0, L_{n-1}]$, all $L_i \geqslant 2$, at least one of the L_i even. The box $\mathcal{D}$ is regular.*

Proof. The proof is by induction on n; Theorem 10.3 serves as the basis of the induction.

Consider $n > 4$ and a box $\mathcal{D}$ as in the statement. For $k \leqslant n - 2$, let $\mathcal{D}_k = [0, L_1] \times \cdots \times [0, L_k L_{k+1}] \times \cdots \times [0, L_{n-1}]$. From Example 10.2 we know that each $\mathcal{D}_k$ can be folded to obtain $\mathcal{D}$. By induction, we know that each $\mathcal{D}_k$ is regular; notice that if $n = 5$ we still have $L_k L_{k+1} > 2$. As in the proof of Theorem 10.3, any domino is compatible with unfolding to all but possibly one $\mathcal{D}_k$. Of course, dominoes which respect the path can be unfolded to any $\mathcal{D}_k$.

We first notice that there exist tilings of twist $1 \in \mathbb{Z}/(2)$ of $\mathcal{D} \times [0, N]$ for some even N. As discussed in Section 7, any tiling $\mathbf{t}$ of $\mathcal{D} \times [0, N]$, N even, is homotopic to a product of tilings $\mathbf{t}_{d;\phi}$ containing a single domino which does not respect the path. Thus, at least one of them has twist 1: call it $\mathbf{t}_1 = \mathbf{t}_{d_1;\phi_1}$. Let $\mathbf{t}_0$ be the vertical tiling of $\mathcal{D} \times [0, 2]$.

We prove that if $\operatorname{Tw}(\mathbf{t}_{d;\phi}) = 0$ then $\mathbf{t}_{d;\phi} \sim \mathbf{t}_0$. Indeed, $\mathbf{t}_{d;\phi}$ can be unfolded to some $\mathcal{D}_k$ to obtain a tiling $\mathbf{t}_2$ of $\mathcal{D}_k \times [0, N]$. By the definition of twist, $\operatorname{Tw}(\mathbf{t}_2) = 0$. Since $\mathcal{D}_k$ is regular, there exists N_2 even and a sequence of flips in $\mathcal{D}_k \times [0, N + N_2]$ taking $\mathbf{t}_2 * \mathbf{t}_{\text{vert},N_2}$ to $\mathbf{t}_{\text{vert},N+N_2}$. Fold this sequence of flips to obtain the desired homotopy in $\mathcal{D}$.

We prove that if $\operatorname{Tw}(\mathbf{t}_{d;\phi}) = 1$ then $\mathbf{t}_{d;\phi} \sim \mathbf{t}_1$. Indeed, d rules out at most one value of k (for unfolding) and d_1 rules out at most another value. There is still at least one value of k such that $\mathbf{t}_{d;\phi} * \mathbf{t}_1^{-1}$ can be unfolded to $\mathcal{D}_k \times [0, N]$. We have $\operatorname{Tw}(\mathbf{t}_{d;\phi} * \mathbf{t}_1^{-1}) = 0$ and therefore, as in the previous paragraph, a sequence of flips in $\mathcal{D}_k$. Fold the sequence as above and we are done. □

12. Proof of Theorem 1.3

We are ready to proceed to the proof of Theorem 1.3. The proof is similar to that of Theorem 2 from [12], but, due to the finiteness of $G_{\mathcal{D}}$, significantly simpler.

Proof of Theorem 1.3. Let $\mathcal{D}$ be a regular region and $\mathcal{C}_{\mathcal{D}}$ be its complex. Let $\Pi : \tilde{\mathcal{C}}_{\mathcal{D}} \to \mathcal{C}_{\mathcal{D}}$ be its universal cover so that $\tilde{\mathcal{C}}_{\mathcal{D}}$ is a simply connected finite complex. Let $\tilde{\mathcal{P}}$ be the finite set of vertices of $\tilde{\mathcal{C}}_{\mathcal{D}}$. Let $\mathbf{p}_\circ \in \tilde{\mathcal{P}}$ be a base point, fixed from now on, satisfying $\Pi(\mathbf{p}_\circ) = \mathbf{p}_\circ \in \mathcal{P}$. A tiling of $\mathcal{R}_{0,N;\mathbf{p}_\circ,p}$ is a walk in $\mathcal{C}_{\mathcal{D}}$ and can therefore be lifted to a continuous path in $\tilde{\mathcal{C}}_{\mathcal{D}}$, starting at $\mathbf{p}_\circ \in \tilde{\mathcal{P}}$ and ending in an element of $\Pi^{-1}[\{p\}] \subset \tilde{\mathcal{P}}$. For each $p \in \tilde{\mathcal{P}}$, let $\mathbf{t}_p$ be a path in $\tilde{\mathcal{C}}_{\mathcal{D}}$ from $\mathbf{p}_\circ$ to p, of length N_p. Assume $\mathbf{t}_{\mathbf{p}_\circ}$ to be the path of length 0 so that $N_{\mathbf{p}_\circ} = 0$.

Let $p_0, p_1 \in \tilde{\mathcal{P}}$. For every floor $\mathbf{f}$ from p_0 to p_1, there exists a homotopy fixing endpoints between $\mathbf{t}_{p_0} * \mathbf{f}$ and $\mathbf{t}_{p_1}$. By construction, there exists an even integer $M_{p_0,p_1,\mathbf{f}} \geqslant \max\{N_{p_0}, N_{p_1}\}$ such that

$$\mathbf{t}_{\text{vert},M_{p_0,p_1,\mathbf{f}}-N_{p_0}} * \mathbf{t}_{p_0} * \mathbf{f} \approx \mathbf{t}_{\text{vert},M_{p_0,p_1,\mathbf{f}}-N_{p_1}} * \mathbf{t}_{p_1}.$$

Let M be even and equal to or larger than the maximum among all $M_{p_0,p_1,\mathbf{f}}$.

Consider a tiling $\mathbf{t}_\dagger$ as a (continuous) path of length $N_\dagger$ in $\tilde{\mathcal{C}}_{\mathcal{D}}$ from $\mathbf{p}_\circ$ to $p_\dagger \in \tilde{\mathcal{P}}$. For $k \in \mathbb{Z}$, $0 \leqslant k \leqslant N_\dagger$, let p_k be the k-th vertex of the path $\mathbf{t}_\dagger$ so that $p_0 = \mathbf{p}_\circ$ and $p_{N_\dagger} = p_\dagger$. Let $\mathbf{t}_{\dagger,k}$ be the restriction of the original path $\mathbf{t}_\dagger$ to $[k, N]$ so that $\mathbf{t}_{\dagger,k} \in \mathcal{T}(\mathcal{R}_{k,N_\dagger;p_k,p_\dagger})$. We construct a homotopy H from $\mathbf{t}_\dagger$ to $\mathbf{t}_{p_\dagger}$. For $k \in \mathbb{Z}$, $0 \leqslant k \leqslant N_\dagger$, set $H(k) = \mathbf{t}_{p_k} * \mathbf{t}_{\dagger,k}$. Notice that $H(0) = \mathbf{t}_\dagger$ and $H(N_\dagger) = \mathbf{t}_{p_\dagger}$. In order to move from $H(k)$ to $H(k+1)$ we proceed as in the previous paragraph, rewriting $H(k) = \mathbf{t}_{p_k} * \mathbf{f} * \mathbf{t}_{\dagger,k+1}$. This step can be accomplished in $\mathcal{R}_{M_{p_k,p_{k+1},\mathbf{f}}+(N_\dagger-k-1)}$. Thus, the entire homotopy can be constructed as a sequence of flips in $\mathcal{R}_{N_\dagger+M}$, completing the proof. ☐

Corollary 12.1. *Let $\mathcal{D} \subset \mathbb{R}^{n-1}$ be a regular region; let M be as in Theorem* 1.3*. Let $\mathbf{t}_0, \mathbf{t}_1$ be tilings of $\mathcal{R}_N$. If both $\mathbf{t}_0$ and $\mathbf{t}_1$ have at least M vertical floors and* $\operatorname{Tw}(\mathbf{t}_0) = \operatorname{Tw}(\mathbf{t}_1)$ *then* $\mathbf{t}_0 \approx \mathbf{t}_1$.

Proof. We know that vertical floors can be moved up and down by flips. In other words, there exist tilings $\mathbf{t}_{0,\bullet}, \mathbf{t}_{1,\bullet}$ of $\mathcal{R}_{N-M}$ with $\mathbf{t}_i \approx \mathbf{t}_{i,\bullet} * \mathbf{t}_{\mathrm{vert},M}$ (for $i \in \{0,1\}$). We also have $\operatorname{Tw}(\mathbf{t}_{i,\bullet}) = \operatorname{Tw}(\mathbf{t}_i)$ and therefore $\operatorname{Tw}(\mathbf{t}_{0,\bullet}) = \operatorname{Tw}(\mathbf{t}_{1,\bullet})$. By regularity, $\mathbf{t}_{0,\bullet} \sim \mathbf{t}_{1,\bullet}$. By Theorem 1.3, $\mathbf{t}_{0,\bullet} * \mathbf{t}_{\mathrm{vert},M} \approx \mathbf{t}_{1,\bullet} * \mathbf{t}_{\mathrm{vert},M}$, as desired. ☐

We now have all the ingredients to prove Corollary 1.4.

Proof of Corollary 1.4. We know that twist partitions $\mathcal{T}(\mathcal{R}_N)$ into two subsets $\tilde{T}_+$ (twist equal to $0 \in \mathbb{Z}/(2)$) and $\tilde{T}_-$ (twist equal to 1). We have $|\tilde{T}_+| - |\tilde{T}_-| = \Delta(\mathcal{R}_N)$. From Lemma 5.3, $|\Delta(\mathcal{R}_N)|$ is exponentially smaller than $|\mathcal{T}(\mathcal{R}_N)|$ (as a function of N).

Let M be as in Corollary 12.1. From Lemma 5.5, for N sufficiently large, most tilings of $\mathcal{R}_N$ admit at least M vertical floors. Let $\mathbf{t}_0, \mathbf{t}_1$ be two such tilings with $\operatorname{Tw}(\mathbf{t}_i) = i \in \mathbb{Z}/(2)$. Let $T_i \subset \mathcal{T}(\mathcal{R}_N)$ be the $\approx$-equivalence class of $\mathbf{t}_i$. We have $T_i \subseteq \tilde{T}_i$. From Corollary 12.1, all tilings which have at least M vertical floors belong to $T_0 \cup T_1$. From Lemma 5.5, $|\mathcal{T}(\mathcal{R}_N) \smallsetminus (T_0 \cup T_1)|/|\mathcal{T}(\mathcal{R}_N)|$ tends to zero exponentially in N. The desired results follow. ☐

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