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ABSTRACT This paper develops the theory of enriched toric $[\vec{D}]$-partitions. Whereas Stembridge's enriched P-partitions give rise to the peak algebra which is a subring of the ring of quasi-symmetric functions QSym, our enriched toric $[\vec{D}]$-partitions generate the cyclic peak algebra which is a subring of the ring of cyclic quasi-symmetric functions cQSym. In the same manner as the peak set of linear permutations appears when considering enriched P-partitions, the cyclic peak set of cyclic permutations plays an important role in our theory. The associated order polynomial is discussed based on this framework.

1. Introduction

Denote by $\mathbb{N}$ and $\mathbb{P}$ the set of nonnegative integers and positive integers respectively. For $m, n \in \mathbb{N}$, define $[m, n] = \{m, m+1, \ldots, n\}$ and write $[n] = [1, n]$ when $m = 1$. A *linear permutation* of a set $A \subset \mathbb{P}$ is an arrangement $w = w_1 w_2 \ldots w_n$ of elements in A where each element is used exactly once. In this case, we call n the *length* of w, written as $\#w = |w| = n$. Let $\mathcal{S}_n$ be the symmetric group on $[n]$ viewed as the set of linear permutations of $[n]$.

A *linear permutation statistic* is a function whose domain is the set of all linear permutations. For a linear permutation $w = w_1 w_2 \ldots w_n$, a *descent* of w is a position i such that $w_i > w_{i+1}$. The *descent set* Des is defined by

$$\operatorname{Des} w = \{i \mid i \text{ is a descent of } w\} \subseteq [n-1].$$

The *descent number* of w is $\operatorname{des} w := |\operatorname{Des} w|$. A *peak* of w is a position i such that $w_{i-1} < w_i > w_{i+1}$. The *peak set* Pk is defined by

$$\operatorname{Pk} w = \{i \mid i \text{ is a peak of } w\} \subseteq [2, n-1].$$

The *peak number* is $\operatorname{pk} w := |\operatorname{Pk} w|$.

Quasi-symmetric functions first appeared implicitly as generating functions in Richard Stanley's theory of P-partitions [9], and then were explicitly studied by Ira M. Gessel [5]. To be more precise, for a finite poset P, the set of P-partitions can be partitioned according to the linear extensions of P, where each subset corresponds to a fundamental quasi-symmetric function indexed by the descent set of that linear permutation. The ring QSym of quasi-symmetric functions was further developed, see [3, 6, 10] for some related articles. It found applications in enumerative combinatorics, representation theory and algebraic geometry [7, 2, 8]. In the same vein,

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Stembridge's work [11] on enriched P-partitions gave rise to the algebra of peaks Π, a graded subring of QSym, which is closely related to the peak set.

For a linear permutation $w = w_1 w_2 \dots w_n$, we define the corresponding *cyclic permutation* $[w]$ to be the set of rotations of w, that is,

$$[w] = \{w_1 w_2 \dots w_n,\ \ w_2 \dots w_n w_1,\ \ \dots,\ \ w_n w_1 \dots w_{n-1}\}.$$

Let $[\mathcal{S}_n]$ denote the set of cyclic permutations on $[n]$.

The *cyclic descent set* cDes of a linear permutation w is defined by

$$\operatorname{cDes} w = \{i \mid w_i > w_{i+1} \text{ where the subscripts are taken modulo } n\} \subseteq [n].$$

This leads to the *cyclic descent set* of a cyclic permutation

$$\operatorname{cDes}[w] = \{\{\operatorname{cDes} \sigma \mid \sigma \in [w]\}\},$$

where the double curly brackets denote a multiset. The multiplicity comes into play since cDes may have the same value on different representatives σ in $[w]$. In fact, one can regard the cyclic permutation statistic cDes as an analogue of Des in the linear setting. Similarly, if we define the *cyclic peak set* cPk of a linear permutation w by

$$\operatorname{cPk} w = \{i \mid w_{i-1} < w_i > w_{i+1} \text{ where the subscripts are taken modulo } n\} \subseteq [n],$$

then the cyclic counterpart of Pk, the *cyclic peak set* cPk of a cyclic permutation, is defined as

$$\operatorname{cPk}[w] = \{\{\operatorname{cPk} \sigma \mid \sigma \in [w]\}\}.$$

Example 1.1. Consider the permutation $w = 3124$ on $[4]$, then

$$\operatorname{Des} w = \{1\},\ \operatorname{cDes} w = \{1, 4\},\ \operatorname{Pk} w = \varnothing,\ \operatorname{cPk} w = \{4\}.$$

Remark 1.2. By definition, $\operatorname{cDes}[w]$ carries the information for all representatives in $[w]$. Moreover, $\operatorname{cDes} w$ together with $|w|$ will be sufficient to determine $\operatorname{cDes}[w]$. In fact, $\operatorname{cDes}[w]$ is simply the multiset of all cyclic shifts of $\operatorname{cDes} w$ in $[n]$ where $n = |w|$, namely,

$$\operatorname{cDes}[w] = \{\{\, i + \operatorname{cDes} w \mid i \in [n] \,\}\}.$$

Here $i + \operatorname{cDes} w$ is the set defined by (1). Similarly, $\operatorname{cPk}[w]$ can be entirely determined by $\operatorname{cPk} w$ and $|w|$.

Example 1.3. Consider the permutation $w = 1423$ and the corresponding cyclic permutation $[w] = \{1423, 3142, 2314, 4231\}$, we have

$$\operatorname{cDes}[w] = \operatorname{cPk}[w] = \{\{\, \{2, 4\}, \{1, 3\}, \{2, 4\}, \{1, 3\} \,\}\}.$$

Noting that $\operatorname{cDes} w = \operatorname{cPk} w = \{2, 4\}$, one can easily check that the previous remark does hold.

In the work [1] of Adin, Gessel, Reiner, and Roichman, the ring cQSym of cyclic quasi-symmetric functions was introduced from toric P-partition enumerators, in which case the cyclic descent set cDes plays an important role. The authors also asked for a cyclic version of the algebra of peaks, to which question we will give an answer in this paper.

This article is devoted to the study of enriched toric $[\vec{D}]$-partitions. By the end, we will construct an algebra of cyclic peaks in cQSym analogous to the algebra of peaks. The rest of this paper is structured as follows. In the next section, we recall definitions of various terms such as quasi-symmetric functions and cyclic quasi-symmetric functions, with several concrete examples provided. Section 3 introduces enriched $\vec{D}$-partitions in terms of directed acyclic graphs (DAGs). In section 4, we define enriched toric $[\vec{D}]$-partitions and develop some of their properties. Section 5 will review the weight enumerators of enriched $\vec{D}$-partitions defined by Stembridge and discuss the cyclic analogues for enriched toric $[\vec{D}]$-partitions. The weight enumerators corresponding to different cyclic peak sets generate a subring of cQSym which we call the algebra of cyclic peaks. We also compute the order polynomial of enriched toric $[\vec{D}]$-partitions.

2. Basic definitions and results

2.1. Sets and compositions. We will use $\leqslant$ with no subscript to denote the ordinary total order on $\mathbb{Z}$, the set of integers. For $n \in \mathbb{P}$, let $2^{[n]}$ denote the set of all subsets of $[n]$, and let $2_0^{[n]}$ be the set of all nonempty subsets of $[n]$. Denote by Comp_n the set of all compositions of n and write $\alpha \vDash n$ for $\alpha \in \mathrm{Comp}_n$. Define a *cyclic shift* of a subset $E \subseteq [n]$ in $[n]$ to be a set of the form

$$i + E = \{i + e \pmod{\mathrm{n}} \mid e \in E\}. \tag{1}$$

Note that sometimes we will use $E + i$ as well for the same concept. While using a negative shift, the reader should be careful to distinguish between $E - i$ and the set difference $E - \{i\} = E \smallsetminus \{i\}$.

A *cyclic shift* of a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a composition of the form

$$(\alpha_k, \dots, \alpha_m, \alpha_1, \dots, \alpha_{k-1})$$

for some $k \in [m]$. We adopt the notations from [1] and denote by $c2_0^{[n]}$ (respectively, cComp_n) the set of equivalence classes of elements of $2_0^{[n]}$ (respectively, Comp_n) under cyclic shifts. Here we recall two natural bijections which will play important roles when indexing two particular bases of (cyclic) quasi-symmetric functions.

The first natural bijection is between $2^{[n-1]}$ and Comp_n. The map $\Phi : 2^{[n-1]} \to \mathrm{Comp}_n$ is defined by

$$\Phi(E) := (e_1 - e_0, e_2 - e_1, \dots, e_k - e_{k-1}, e_{k+1} - e_k) \tag{2}$$

for any given $E = \{e_1 < e_2 < \cdots < e_k\} \subseteq [n-1]$ with $e_0 = 0$ and $e_{k+1} = n$, where the inverse map is

$$\Phi^{-1}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_k\}$$

for any $\alpha = (\alpha_1, \dots, \alpha_{k+1}) \vDash n$.

Another bijection is between $c2_0^{[n]}$ and cComp_n, for the sake of which we need to consider the map $\psi : 2_0^{[n]} \to \mathrm{Comp}_n$ defined by

$$\psi(E) := (e_2 - e_1, \dots, e_k - e_{k-1}, e_1 - e_k + n) \tag{3}$$

where $E = \{e_1 < e_2 < \cdots < e_k\} \subseteq [n]$. Notice that if E' is a cyclic shift of E in $[n]$, then $\psi(E')$ is also a cyclic shift of $\psi(E)$. So ψ induces a map $\Psi : c2_0^{[n]} \to \mathrm{cComp}_n$. Moreover, it is straightforward to check that the induced map Ψ is bijective.

2.2. Quasi-symmetric functions QSym. A *quasi-symmetric function* is a formal power series $f \in \mathbb{Q}[[x_1, x_2, \dots]]$ such that for any sequence of positive integers $a = (a_1, a_2, \dots, a_s)$, and two increasing sequences $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_s$ of positive integers,

$$[x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_s}^{a_s}]\, f = [x_{j_1}^{a_1} x_{j_2}^{a_2} \dots x_{j_s}^{a_s}]\, f,$$

where $[x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_s}^{a_s}]\, f$ denotes the coefficient of monomial $x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_s}^{a_s}$ in the expression of f. Let QSym_n be the set of all quasi-symmetric functions which are homogeneous of degree n, and $\mathrm{QSym} = \oplus_{n \geqslant 0} \mathrm{QSym}_n$. Two bases of QSym are particularly important to our work: monomial quasi-symmetric functions M_L and fundamental quasi-symmetric functions F_L.

Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \vDash n$, the associated *monomial quasi-symmetric function* indexed by α is

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_s} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_s}^{\alpha_s}.$$

The $\{M_\alpha\}_{\alpha \vDash n}$ form a basis of QSym_n. From the bijection $\Phi : 2^{[n-1]} \to \mathrm{Comp}_n$ defined by (2), we can also index the monomial quasi-symmetric functions by subsets $E \subseteq [n-1]$, and define $M_{n,E} := M_{\Phi(E)}$.

There is another important basis of QSym. The *fundamental quasi-symmetric function* indexed by $E \subseteq [n-1]$ is

$$F_{n,E} = \sum_{\substack{i_1 \leqslant \cdots \leqslant i_n \\ i_k < i_{k+1} \text{ if } k \in E}} x_{i_1} x_{i_2} \dots x_{i_n}.$$

Similarly, we can define $F_\alpha := F_{\Phi^{-1}(\alpha)}$ indexed by compositions.

The relation between monomial and fundamental quasi-symmetric functions is simple:

$$F_{n,E} = \sum_{L \supseteq E} M_{n,L}. \tag{4}$$

By the principle of inclusion and exclusion, $M_{n,E}$ can be expressed as a linear combination of the $F_{n,L}$, from which we can tell that $\{F_{n,L}\}_{L \subseteq [n-1]}$ spans QSym_n. By checking the cardinality of both sets $\{F_{n,L}\}_{L \subseteq [n-1]}$ and $\{M_{n,E}\}_{E \subseteq [n-1]}$, it follows that $\{F_{n,L}\}_{L \subseteq [n-1]}$ is indeed a basis of QSym_n.

Example 2.1. Consider $E = \{1, 3\}$, by definition we have

$$F_{4,\{1,3\}} = \sum_{i_1 < i_2 \leqslant i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} = \sum_{i_1 < i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} + \sum_{i_1 < i_2 < i_4} x_{i_1} x_{i_2}^2 x_{i_4}.$$

There are only two choices for a set L satisfying $E \subseteq L \subseteq [3]$: $\{1, 3\}$ or $\{1, 2, 3\}$. Since $\Phi(\{1, 3\}) = (1, 2, 1), \Phi(\{1, 2, 3\}) = (1, 1, 1, 1)$, we get

$$M_{4,\{1,3\}} = M_{(1,2,1)} = \sum_{i_1 < i_2 < i_3} x_{i_1} x_{i_2}^2 x_{i_3}, \;\; M_{4,\{1,2,3\}} = M_{(1,1,1,1)} = \sum_{i_1 < i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}.$$

The calculation above verifies that $F_{4,\{1,3\}} = M_{4,\{1,3\}} + M_{4,\{1,2,3\}} = \sum_{L \supseteq \{1,3\}} M_{4,L}$.

2.3. Cyclic quasi-symmetric functions cQSym. In this subsection, we recall from [1] the theory of cyclic quasi-symmetric functions. We will model our work of enriched toric $[\vec{D}]$-partitions with enumerators in this environment.

A *cyclic quasi-symmetric function* is a formal power series $f \in \mathbb{Q}[[x_1, x_2, \dots]]$ such that for any sequence of positive integers $a = (a_1, a_2, \dots, a_s)$, a cyclic shift $(a_1', a_2', \dots, a_s')$ of a, and two increasing sequences $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_s$ of positive integers,

$$[x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_s}^{a_s}]\, f = [x_{j_1}^{a_1'} x_{j_2}^{a_2'} \dots x_{j_s}^{a_s'}]\, f,$$

namely the coefficients of $x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_s}^{a_s}$ and $x_{j_1}^{a_1'} x_{j_2}^{a_2'} \dots x_{j_s}^{a_s'}$ in f are equal. Denote by cQSym_n the set of all cyclic quasi-symmetric functions which are homogeneous of degree n, and $\mathrm{cQSym} = \oplus_{n \geqslant 0} \mathrm{cQSym}_n$.

Remark 2.2. It is clear that there exists a strict inclusion relation $\mathrm{Sym} \subsetneq \mathrm{cQSym} \subsetneq \mathrm{QSym}$, where Sym is the algebra of symmetric functions.

$\alpha \vDash 4$	$M^{\mathrm{cyc}}_{\alpha}$
(4)	$M_{(4)} = M_{4,\varnothing}$
(1,3) or (3,1)	$M_{(1,3)} + M_{(3,1)} = M_{4,\{1\}} + M_{4,\{3\}}$
$(2,2)$	$2M_{(2,2)} = 2M_{4,\{2\}}$
(1,1,2) or (1,2,1) or (2,1,1)	$M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,1,1)} =$ $M_{4,\{1,2\}} + M_{4,\{1,3\}} + M_{4,\{2,3\}}$
$(1,1,1,1)$	$4M_{(1,1,1,1)} = 4M_{4,\{1,2,3\}}$

Table 1. Monomial cyclic quasi-symmetric functions indexed by compositions of 4

We have the following cyclic analogues of the concepts of monomial (fundamental) quasi-symmetric functions.

Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \vDash n$, the associated *monomial cyclic quasi-symmetric function* indexed by α is

$$M^{\mathrm{cyc}}_{\alpha} = \sum_{i=1}^{s} M_{(\alpha_i, \alpha_{i+1}, \dots, \alpha_{i-1})},$$

where the indices are interpreted modulo s, meaning $\alpha_j = \alpha_{j+s}$. In other words, $M^{\mathrm{cyc}}_{\alpha}$ sums over all monomial quasi-symmetric functions indexed by cyclic shifts of α. Therefore it is clear that $M^{\mathrm{cyc}}_{\alpha} = M^{\mathrm{cyc}}_{\alpha'}$ if α and α' only differ by a cyclic shift.

We can also index the monomial cyclic quasi-symmetric function by sets. For a nonempty $E \subseteq [n]$, define $M^{\mathrm{cyc}}_{n,E} := M^{\mathrm{cyc}}_{\psi(E)}$ via the map $\psi : 2^{[n]}_0 \to \mathrm{Comp}_n$ defined by (3), and set $M^{\mathrm{cyc}}_{n,\varnothing} := 0$. Similarly it can be shown that $M^{\mathrm{cyc}}_{n,E} = M^{\mathrm{cyc}}_{n,E'}$ if E' is a cyclic shift of E.

The following result gives the expression of monomial cyclic quasi-symmetric functions in terms of monomial quasi-symmetric functions.

Lemma 2.3 ([1, Lemma 2.5], monomial to cyclic monomial). *For any subset $E \subseteq [n]$*

$$M^{\mathrm{cyc}}_{n,E} = \sum_{e \in E} M_{n,(E-e)\cap[n-1]}, \tag{5}$$

where the set $E - e$ is defined as (1).

Example 2.4. Table 1 computes all monomial cyclic quasi-symmetric function indexed by compositions of 4, in terms of monomial quasi-symmetric functions.

For the natural desire of establishing a cyclic analogue of the relation between monomial and fundamental quasi-symmetric functions given by (4), define the *fundamental cyclic quasi-symmetric function* indexed by $E \subseteq [n]$ as

$$F^{\mathrm{cyc}}_{n,E} := \sum_{L \supseteq E} M^{\mathrm{cyc}}_{n,L}. \tag{6}$$

Remark 2.5. This is not the original definition in [1] but appears as a lemma in the same article. But these two definitions are equivalent via [1, Lemma 2.14]. We will use this definition for the purpose of our work, and mention the original definition in Proposition 2.11 for interested readers.

Example 2.6. Consider $n = 4$ and $E = \{1, 3\}$. By definition

$$F^{\mathrm{cyc}}_{4,\{1,3\}} = \sum_{L \supseteq \{1,3\}} M^{\mathrm{cyc}}_{4,L}$$

$$
\begin{aligned}
&\overset{(i)}{=} \quad M^{\mathrm{cyc}}_{4,\{1,3\}} + M^{\mathrm{cyc}}_{4,\{1,2,3\}} + M^{\mathrm{cyc}}_{4,\{1,3,4\}} + M^{\mathrm{cyc}}_{4,\{1,2,3,4\}} \\
&\overset{(ii)}{=} \quad M^{\mathrm{cyc}}_{(2,2)} + M^{\mathrm{cyc}}_{(1,1,2)} + M^{\mathrm{cyc}}_{(2,1,1)} + M^{\mathrm{cyc}}_{(1,1,1,1)} \\
&\overset{(iii)}{=} \quad 2M_{(2,2)} + 2\left(M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,1,1)}\right) + 4M_{(1,1,1,1)} \\
&= \quad 2\sum_{i_1<i_2} x_{i_1}^2 x_{i_2}^2 \\
&\qquad + 2\sum_{i_1<i_2<i_3} (x_{i_1}x_{i_2}x_{i_3}^2 + x_{i_1}x_{i_2}^2x_{i_3} + x_{i_1}^2x_{i_2}x_{i_3}) \\
&\qquad + 4\sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4}.
\end{aligned}
$$

Equality (i) follows from the fact that the choices for $L \supseteq \{1,3\}$ in $[4]$ are $\{1,3\}$, $\{1,2,3\}$, $\{1,3,4\}$ and $\{1,2,3,4\}$. Equality (ii) is obtained by changing indices under the map ψ defined by (3), equality (iii) is from Table 1.

The following transition from fundamental to cyclic fundamental quasi-symmetric functions should come without surprise.

Lemma 2.7 ([1, Proposition 2.15], fundamental to cyclic fundamental). *For any subset* $E \subseteq [n]$,

$$F^{\mathrm{cyc}}_{n,E} = \sum_{i\in[n]} F_{n,(E-i)\cap[n-1]},$$

with set $E-i$ *defined by* (1).

Remark 2.8.

(1) It follows directly from Lemma 2.7 that $F^{\mathrm{cyc}}_{n,E} = F^{\mathrm{cyc}}_{n,E'}$ if E' is a cyclic shift of E.

(2) Clearly the set $\{M^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$ spans cQSym_n and is linearly independent, as each monomial of degree n appears in $M^{\mathrm{cyc}}_{n,E}$ for exactly one $E \in c2^{[n]}_0$. Hence $\{M^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$ is a basis of cQSym_n. Applying the principle of inclusion and exclusion on (6) we have

$$M^{\mathrm{cyc}}_{n,E} = \sum_{L\supseteq E} (-1)^{|L\smallsetminus E|} F^{\mathrm{cyc}}_{n,L},$$

which implies that $\{F^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$ also spans cQSym_n; together with the fact that the dimension of vector space cQSym_n is $\#c2^{[n]}_0$, $\{F^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$ is also a basis of cQSym_n.

Now we review the original definition of cyclic quasi-symmetric functions. Before that, we need to associate with each $E \subseteq [n]$ a set $P^{\mathrm{cyc}}_{n,E}$.

Definition 2.9. *Let* $P^{\mathrm{cyc}}_{n,E}$ *denote the set of all pairs* (w,k) *where* $w = w_1 \dots w_n$ *is a sequence of positive integers and* $k \in [n]$ *satisfying*

(1) *w is "cyclically weakly increasing" from index k, that is,* $w_k \leqslant w_{k+1} \leqslant \dots \leqslant w_n \leqslant w_1 \leqslant \dots \leqslant w_{k-1}$.

(2) *If* $i \in E \smallsetminus \{k-1\}$, *then* $w_i < w_{i+1}$, *where the indices are computed modulo* n.

Remark 2.10. The index k is uniquely defined by w unless all integers in w are the same. In that case, either $E = \{k-1\}$ for some $k \in [n]$ or $E = \varnothing$. Those w with elements all equal will only be paired with $k-1$ if $E = \{k-1\}$, and so get counted once in $P^{\mathrm{cyc}}_{n,E}$; as for $E = \varnothing$, if w has all elements equal, it can pair with every $k \in [n]$, therefore it will be counted n times in $P^{\mathrm{cyc}}_{n,E}$.

PROPOSITION 2.11. *For every subset $E \subseteq [n]$,*

$$F^{\mathrm{cyc}}_{n,E} = \sum_{(w,k)\in P^{\mathrm{cyc}}_{n,E}} x_{w_1}x_{w_2}\dots x_{w_n}.$$

Proof. The proof is essentially the same as for [1, Lemma 2.14]. □

Here we provide a concrete example to illustrate the validity of the proposition above.

EXAMPLE 2.12. Consider $n = 4$ and $E = \{1,3\}$. It is clear that $P^{\mathrm{cyc}}_{n,E}$ has a natural partition into four parts where each part contains all pairs with a given index $k \in [4]$. This implies the following summation

$$\begin{aligned}\sum_{(w,k)\in P^{\mathrm{cyc}}_{4,\{1,3\}}} x_{w_1}x_{w_2}x_{w_3}x_{w_4} &= \sum_{k=1}^{4}\sum_{(w,k)\in P^{\mathrm{cyc}}_{4,\{1,3\}}} x_{w_1}x_{w_2}x_{w_3}x_{w_4}\\ &= \sum_{w_1<w_2\leqslant w_3<w_4} x_{w_1}x_{w_2}x_{w_3}x_{w_4} + \sum_{w_2\leqslant w_3<w_4\leqslant w_1} x_{w_1}x_{w_2}x_{w_3}x_{w_4}\\ &\quad + \sum_{w_3<w_4\leqslant w_1<w_2} x_{w_1}x_{w_2}x_{w_3}x_{w_4} + \sum_{w_4\leqslant w_1<w_2\leqslant w_3} x_{w_1}x_{w_2}x_{w_3}x_{w_4},\end{aligned}$$

For each summand,

$$\begin{aligned}\sum_{w_1<w_2\leqslant w_3<w_4} x_{w_1}x_{w_2}x_{w_3}x_{w_4} &= \sum_{w_1<w_2<w_3<w_4} x_{w_1}x_{w_2}x_{w_3}x_{w_4} + \sum_{w_1<w_2=w_3<w_4} x_{w_1}x_{w_2}x_{w_3}x_{w_4}\\ &= \sum_{w_1<w_2<w_3<w_4} x_{w_1}x_{w_2}x_{w_3}x_{w_4} + \sum_{w_1<w_2<w_3} x_{w_1}x_{w_2}^2x_{w_3};\end{aligned}$$

$$\begin{aligned}\sum_{w_2\leqslant w_3<w_4\leqslant w_1} x_{w_1}x_{w_2}x_{w_3}x_{w_4} &= \sum_{w_2<w_3<w_4<w_1} x_{w_2}x_{w_3}x_{w_4}x_{w_1} + \sum_{w_2=w_3<w_4<w_1} x_{w_2}x_{w_3}x_{w_4}x_{w_1}\\ &\quad + \sum_{w_2<w_3<w_4=w_1} x_{w_2}x_{w_3}x_{w_4}x_{w_1} + \sum_{w_2=w_3<w_4=w_1} x_{w_2}x_{w_3}x_{w_4}x_{w_1}\\ &= \sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4} + \sum_{i_1<i_2<i_3} x_{i_1}^2x_{i_2}x_{i_3}\\ &\quad + \sum_{i_1<i_2<i_3} x_{i_1}x_{i_2}x_{i_3}^2 + \sum_{i_1<i_2} x_{i_1}^2x_{i_2}^2;\end{aligned}$$

$$\begin{aligned}\sum_{w_3<w_4\leqslant w_1<w_2} x_{w_1}x_{w_2}x_{w_3}x_{w_4} &= \sum_{w_3<w_4<w_1<w_2} x_{w_3}x_{w_4}x_{w_1}x_{w_2} + \sum_{w_3<w_4=w_1<w_2} x_{w_3}x_{w_4}x_{w_1}x_{w_2}\\ &= \sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4} + \sum_{w_1<w_2<w_3} x_{w_1}x_{w_2}^2x_{w_3};\end{aligned}$$

$$\begin{aligned}\sum_{w_4\leqslant w_1<w_2\leqslant w_3} x_{w_1}x_{w_2}x_{w_3}x_{w_4} &= \sum_{w_4<w_1<w_2<w_3} x_{w_4}x_{w_1}x_{w_2}x_{w_3} + \sum_{w_4=w_1<w_2<w_3} x_{w_4}x_{w_1}x_{w_2}x_{w_3}\\ &\quad + \sum_{w_4<w_1<w_2=w_3} x_{w_4}x_{w_1}x_{w_2}x_{w_3} + \sum_{w_4=w_1<w_2=w_3} x_{w_4}x_{w_1}x_{w_2}x_{w_3}\\ &= \sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4} + \sum_{i_1<i_2<i_3} x_{i_1}^2x_{i_2}x_{i_3}\\ &\quad + \sum_{i_1<i_2<i_3} x_{i_1}x_{i_2}x_{i_3}^2 + \sum_{i_1<i_2} x_{i_1}^2x_{i_2}^2.\end{aligned}$$

To sum up, we have

$$\sum_{(w,k)\in P^{\mathrm{cyc}}_{4,\{1,3\}}} x_{w_1}x_{w_2}x_{w_3}x_{w_4} = 4\sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4} + 2\sum_{i_1<i_2<i_3}(x_{i_1}^2x_{i_2}x_{i_3}+x_{i_1}x_{i_2}^2x_{i_3}+x_{i_1}x_{i_2}x_{i_3}^2) + 2\sum_{i_1<i_2} x_{i_1}^2x_{i_2}^2, \tag{7}$$

which is exactly the expression of $F^{\mathrm{cyc}}_{4,\{1,3\}}$ we obtained in Example 2.6.

3. Enriched $\vec{D}$-partitions for DAGs

In [11], Stembridge defined enriched P-partitions of a poset P. In contrast to having $\mathbb{P}$ with the ordinary order $\leqslant$ as the range for ordinary P-partitions, enriched P-partitions are obtained by imposing another total order on the range, $\mathbb{P}'$, which is the set of nonzero integers with an unusual order. We review the basic theory from Stembridge and naturally extend enriched P-partitions to enriched $\vec{D}$-partitions where $\vec{D}$ is a directed acyclic graph which is not necessarily transitive.

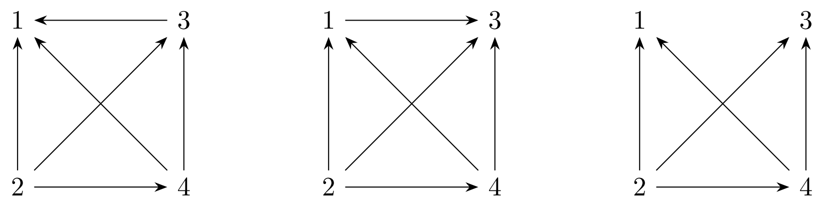
A *directed acyclic graph* (DAG) is a digraph with no directed cycles. Suppose $\vec{D}$ is a DAG with vertex set $[n]$. A DAG $\vec{D}$ is *transitive* if $i\to j$ and $j\to k$ implies $i\to k$ for $i,j,k\in[n]$. If $\vec{P}$ is a transitive DAG, it will naturally induce a partial order $<_{\vec{P}}$ on the vertex set $[n]$, defined so that for two vertices i and j, one has $i<_{\vec{P}} j$ if and only if $i\to j$ in $\vec{P}$; in that case, we also use $\vec{P}$ to denote this poset.

In general, we can associate a partial order with any DAG $\vec{D}$. For this purpose, define the *transitive closure* $\vec{P}$ of $\vec{D}$ as the directed graph obtained from $\vec{D}$ by adding in $i\to k$ if one has both $i\to j$ and $j\to k$ in $\vec{D}$. Such $\vec{P}$ is unique. Moreover, it is straightforward to verify that the transitive closure $\vec{P}$ is both acyclic and transitive. This implies that $\vec{P}$ is actually a transitive DAG, hence has a partial order structure. We will maintain the use $\to$ for the relation between vertices in a general DAG $\vec{D}$, while using the partial order $\leqslant_{\vec{P}}$ on a poset $\vec{P}$.

A poset $\vec{P}$ is a *total linear order* if it is a complete DAG, i.e., there is a directed edge between every pair of vertices in $\vec{P}$. There is a bijection between the set of total linear orders $\vec{P}$ on the vertex set $[n]$ and $\mathcal{S}_n$. For a total linear order $\vec{P}$ on $[n]$, there exists a unique directed path $w_1\to w_2\to\cdots\to w_n$ in $\vec{P}$, hence $\vec{P}$ can be identified with the permutation $w=w_1w_2\dots w_n\in\mathcal{S}_n$. Conversely, given a permutation $w=w_1w_2\dots w_n\in\mathcal{S}_n$, we can construct a DAG $\vec{P}$ by putting arrows $w_i\to w_j$ for all $1\leqslant i<j\leqslant n$ on the vertex set $[n]$, and it is easy to check that the resulting DAG $\vec{P}$ is actually a total linear order. In this case, we usually use a permutation w to denote the corresponding total linear order $\vec{P}$.

For two DAGs $\vec{D}_1$ and $\vec{D}_2$ on the same vertex set $[n]$, we say $\vec{D}_2$ *extends* $\vec{D}_1$ if $\vec{D}_1$ is a subgraph of $\vec{D}_2$, written as $\vec{D}_1\subseteq\vec{D}_2$. If, furthermore, $\vec{D}_2$ is a total linear order corresponding to the permutation $w\in\mathcal{S}_n$, we also say that w *linearly extends* $\vec{D}_1$. Denote by $\mathcal{L}(\vec{D})$ the set of all permutations $w\in\mathcal{S}_n$ which linearly extend $\vec{D}$.

Example 3.1. Here are several DAGs on the vertex set $[4]=\{1,2,3,4\}$.



$\vec{D}_1 = 2431$ $\qquad$ $\vec{D}_2 = 2413$ $\qquad$ $\vec{D}_3$

Note that $\vec{D}_1$ is the total linear order with permutation 2431, $\vec{D}_2$ is the total linear order for 2413. Both $\vec{D}_1$ and $\vec{D}_2$ extend $\vec{D}_3$, hence 2431 and 2413 linearly extend $\vec{D}_3$. Moreover, $\vec{D}_1$ and $\vec{D}_2$ are the only total linear orders which extend $\vec{D}_3$, and therefore $\mathcal{L}(\vec{D}_3) = \{2431, 2413\}$.

Now we are in a good position to define enriched $\vec{D}$-partitions for a DAG $\vec{D}$. Stembridge originally defined the enriched P-partitions when P is a poset. This definition can be easily extended to the cases when $\vec{D}$ is simply a DAG, as the definition does not rely on the transitivity of P.

Stembridge defines $\mathbb{P}'$ to be the set of nonzero integers, totally ordered as

$$-1 \prec 1 \prec -2 \prec 2 \prec -3 \prec 3 \prec \cdots .$$

Definition 3.2 (Enriched $\vec{D}$-partition). *Let $\vec{D}$ be a directed acyclic graph (DAG) on $[n]$. An* enriched $\vec{D}$-partition *is a function $f : [n] \to \mathbb{P}'$ such that for all $i \to j$ in $\vec{D}$,*

(a) $f(i) \preceq f(j)$,
(b) $f(i) = f(j) > 0$ *implies* $i < j$,
(c) $f(i) = f(j) < 0$ *implies* $i > j$.

Denote by $\mathcal{E}(\vec{D})$ the set of all enriched $\vec{D}$-partitions f.

Remark 3.3.

(1) In this definition we are using two order structures on the domain $[n]$: the order $\to$ induced by DAG $\vec{D}$ and the ordinary total order $\leqslant$ on integers in (b) and (c). Both of them will impose restrictions on the possible choices for f. As for the range $\mathbb{P}'$, we also use two order structures: the total order $\preceq$ defined by Stembridge in (a) and the usual order $\leqslant$ on the integers in (b) and (c).
(2) If $\vec{D} = w$ is a total linear order, the structure of the set of enriched w-partitions is quite simple:

$$\begin{aligned} \mathcal{E}(w) = \{\, f : [n] \to \mathbb{P}' \mid {} & f(w_1) \preceq \cdots \preceq f(w_n), \\ & f(w_i) = f(w_{i+1}) > 0 \Rightarrow i \notin \mathrm{Des}(w), \\ & f(w_i) = f(w_{i+1}) < 0 \Rightarrow i \in \mathrm{Des}(w) \,\}. \end{aligned} \tag{8}$$

The following fundamental lemma is a straightforward analogue of Stembridge [11, Lemma 2.1].

Lemma 3.4 (Fundamental lemma of enriched $\vec{D}$-partitions). *For any DAG $\vec{D}$ with vertex set $[n]$, one has a decomposition of $\mathcal{E}(\vec{D})$ as the following disjoint union:*

$$\mathcal{E}(\vec{D}) = \bigsqcup_{w \in \mathcal{L}(\vec{D})} \mathcal{E}(w).$$

Proof. Given an enriched $\vec{D}$-partition f. First we arrange the elements of $[n]$ in a weakly increasing order of f-values with respect to the total order $\preceq$ on the range. Then if some elements in $[n]$ have the same f-value $-k$ (respectively, $+k$) for some positive integer k, we arrange them in a decreasing (respectively, increasing) order with respect to the usual order $\leqslant$ on the domain. The resulting permutation w is unique with $f \in \mathcal{E}(w)$, and w linearly extends $\vec{D}$. On the other hand, for $w \in \mathcal{L}(\vec{D})$, every enriched w-partition is also an enriched $\vec{D}$-partition. Therefore the conclusion follows. $\square$

Example 3.5. Returning to Example 3.1, since $\mathcal{L}(\vec{D}_3) = \{2431, 2413\}$, by the Fundamental Lemma, $\mathcal{E}(\vec{D}_3) = \mathcal{E}(2431) \uplus \mathcal{E}(2413)$.

4. Enriched toric $[\vec{D}]$-partitions for toric DAGs

In this section, we review the toric DAGs and toric posets as cyclic analogues of DAGs and posets. Then we define enriched toric $[\vec{D}]$-partitions and develop some of their properties. The concept of toric poset was originally defined and studied by Develin, Macauley and Reiner in [4]. Here we follow the presentation from Adin, Gessel, Reiner and Roichman [1].

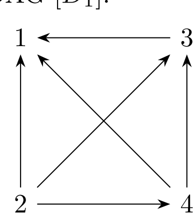
Just like a linear permutation has a corresponding cyclic permutation as the equivalence class under the equivalence of rotation, for a DAG, we will define an equivalence relation and consider the equivalence class to be the corresponding toric DAG. It turns out that if w is a linear extension of the DAG $\vec{D}$, then $[w]$ is a toric extension of the corresponding toric DAG $[\vec{D}]$.

A DAG $\vec{D}$ on $[n]$ has $i_0 \in [n]$ as a *source* (respectively, *sink*) if $\vec{D}$ does not contain $j \to i_0$ (respectively, $i_0 \to j$) for any $j \in [n]$. Suppose i_0 is a source or a sink in $\vec{D}$, we say $\vec{D}'$ is obtained from $\vec{D}$ by a *flip* at i_0 if $\vec{D}'$ is obtained by reversing all arrows containing i_0. We define the equivalence relation $\equiv$ on DAGs as follows: $\vec{D}' \equiv \vec{D}$ if and only if $\vec{D}'$ is obtained from $\vec{D}$ by a sequence of flips. A *toric* DAG is the equivalence class $[\vec{D}]$ of a DAG $\vec{D}$.

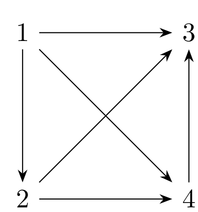
In particular, if $\vec{D} = w = w_1 w_2 \ldots w_n$ is a total linear order, the next proposition claims that the corresponding toric DAG $[\vec{D}]$ can be identified with the cyclic permutation $[w]$.

Proposition 4.1 ([4, Proposition 4.2]). *If $\vec{D} = w$ is a total linear order with $w = w_1 \ldots w_n$, then there is a bijection between toric DAG $[\vec{D}]$ and cyclic permutation $[w]$.*

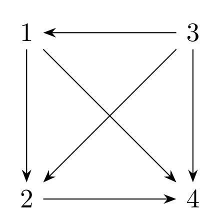
Example 4.2. The total linear order $\vec{D}_1 = 2431$ from Example 3.1 has a corresponding toric DAG $[\vec{D}_1]$:



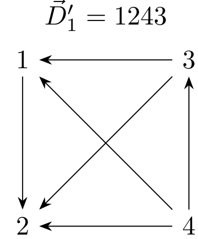
$\vec{D}_1 = 2431$



$\vec{D}_1' = 1243$



$\vec{D}_1'' = 3124$



$\vec{D}_1''' = 4312$

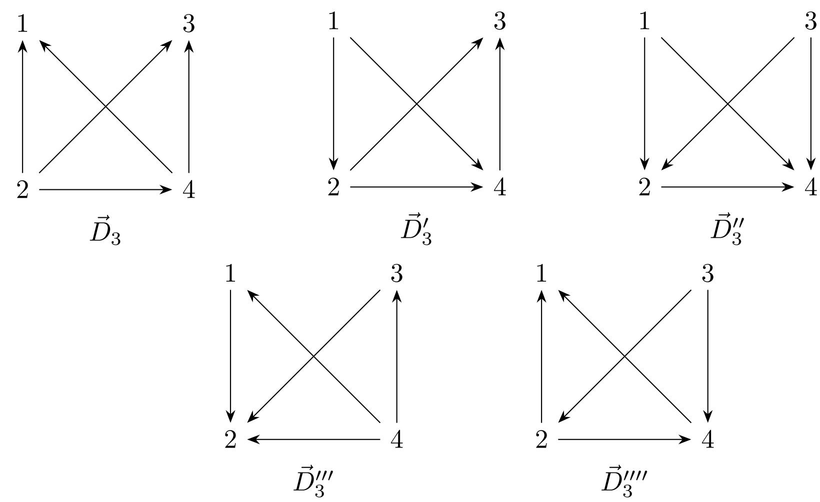


FIGURE 1. All representatives of $[\vec{D}_3]$

They can be obtained by a sequence of flips:

$$\vec{D}_1 \xrightarrow{\text{flip at }1} \vec{D}_1' \xrightarrow{\text{flip at }3} \vec{D}_1'' \xrightarrow{\text{flip at }4} \vec{D}_1''' \xrightarrow{\text{flip at }2} \vec{D}_1.$$

Therefore, $[\vec{D}_1]$ can be identified with $[2431] = \{\, 2431, 1243, 3124, 4312 \,\}$.

As transitivity turns a DAG into a poset, we now introduce the definition of toric transitivity for a DAG, which will turn the corresponding toric DAG into a toric poset.

A DAG $\vec{D}$ with vertex set $[n]$ is *toric transitive* if the existence of a directed path $i_1 \to i_2 \to \cdots \to i_k$ and $i_1 \to i_k$ implies the existence of $i_a \to i_b$ in $\vec{D}$ for all $1 \leqslant a < b \leqslant k$. A toric DAG $[\vec{D}]$ is a *toric poset* if $\vec{D}'$ is toric transitive for some $\vec{D}' \in [\vec{D}]$, or equivalently from [1, Proposition 3.10], for all representatives $\vec{D}'$. A toric poset is a *total cyclic order* if one (or equivalently, all, according to Proposition 4.1) representative is a total linear order. In this case, we usually use the corresponding cyclic permutation $[w]$ to denote the total cyclic order $[\vec{D}]$.

REMARK 4.3. In this paper, we adopt the definition of toric posets from [1], which is not quite the same as it was originally defined in [4], but they are essentially equivalent by [4, Theorem 1.4].

For two toric DAGs $[\vec{D}_1], [\vec{D}_2]$ on the same vertex set $[n]$, we say $[\vec{D}_2]$ *extends* $[\vec{D}_1]$ if there exist $\vec{D}_i' \in [\vec{D}_i']$ for $i = 1, 2$ such that $\vec{D}_2'$ extends $\vec{D}_1'$. If, furthermore, $[\vec{D}_2]$ is a total cyclic order corresponding to the cyclic permutation $[w]$, we also say that $[w]$ torically extends $[\vec{D}_1]$. Let $\mathcal{L}^{\mathrm{tor}}([\vec{D}])$ denote the set of cyclic permutations $[w]$ which torically extend $[\vec{D}]$.

EXAMPLE 4.4. In Example 3.1, both $[\vec{D}_1] = [2431]$ and $[\vec{D}_2] = [2413]$ torically extend $[\vec{D}_3]$. Moreover, they are the only total cyclic orders that torically extend $[\vec{D}_3]$, namely $\mathcal{L}^{\mathrm{tor}}([\vec{D}_3]) = \{[2431], [2413]\}$.

In fact, Figure 1 lists all representatives of $[\vec{D}_3]$, and it is straightforward to check that every total linear order linearly extending some DAG in $[\vec{D}_3]$ is in either [2431] or [2413].

Definition 4.5 (Enriched toric $[\vec{D}]$-partition). *An* enriched toric $[\vec{D}]$-partition *is a function $f : [n] \to \mathbb{P}'$ which is an enriched $\vec{D}'$-partition for at least one DAG $\vec{D}'$ in $[\vec{D}]$. Let $\mathcal{E}^{\mathrm{tor}}([\vec{D}])$ denote the set of all enriched toric $[\vec{D}]$-partitions.*

If $[\vec{D}] = [w]$ is a total cyclic order, the set of enriched toric $[w]$-partitions is the union of the set of enriched w'-partitions for all representatives w' of $[w]$:

$$\mathcal{E}^{\mathrm{tor}}([w]) = \bigcup_{w' \in [w]} \mathcal{E}(w'). \tag{9}$$

As in the linear case, we have the following fundamental lemma for the decomposition of enriched toric $[\vec{D}]$-partitions. The proof is analogous to [1, Lemma 3.15].

Lemma 4.6 (Fundamental lemma of enriched toric $[\vec{D}]$-partitions). *For a DAG $\vec{D}$, the set of all enriched toric $[\vec{D}]$-partitions is a disjoint union of the set of enriched toric $[w]$-partitions of all toric extensions $[w]$ of $[\vec{D}]$:*

$$\mathcal{E}^{\mathrm{tor}}([\vec{D}]) = \bigsqcup_{[w] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])} \mathcal{E}^{tor}([w]).$$

Proof. By the definition of $\mathcal{E}^{\mathrm{tor}}([\vec{D}])$, one has

$$\mathcal{E}^{\mathrm{tor}}([\vec{D}]) = \bigcup_{\vec{D}' \in [\vec{D}]} \mathcal{E}(\vec{D}').$$

In particular when $[\vec{D}] = [w]$ is a total cyclic order, it follows from Proposition 4.1 that,

$$\mathcal{E}^{\mathrm{tor}}([w]) = \bigcup_{w' \in [w]} \mathcal{E}(w').$$

Hence,

$$\begin{aligned} \mathcal{E}^{\mathrm{tor}}([\vec{D}]) &= \bigcup_{\vec{D}' \in [\vec{D}]} \mathcal{E}(\vec{D}') \\ &\overset{(i)}{=} \bigcup_{\vec{D}' \in [\vec{D}]} \bigcup_{w' \in \mathcal{L}(\vec{D}')} \mathcal{E}(w') \\ &\overset{(ii)}{=} \bigcup_{[w] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])} \bigcup_{w' \in [w]} \mathcal{E}(w') \\ &= \bigcup_{[w] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])} \mathcal{E}^{\mathrm{tor}}([w]). \end{aligned}$$

To justify these steps, first note that equality (i) follows from Lemma 3.4.

As for equality (ii), it suffices to show that $w' \in \mathcal{L}(\vec{D}')$ for some $\vec{D}' \in [\vec{D}]$ if and only if $w' \in [w]$ for some $[w] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])$. For the forward direction, if $w' \in \mathcal{L}(\vec{D}')$ for some $\vec{D}' \in [\vec{D}]$, then $[w']$ torically extends $[\vec{D}'] = [\vec{D}]$. For the reverse implication, given $w' \in [w] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])$, pick $\vec{D}'' \in [\vec{D}]$ and $w'' \in [w]$ with w'' linearly extending $\vec{D}''$, then $[w''] = [w'] = [w]$. It follows that there exists a sequence of flips which takes w'' to w'. Now applying the same sequence of flips to $\vec{D}''$ will result in some $\vec{D}'$. One then has $\vec{D}' \in [\vec{D}''] = [\vec{D}]$ and $w' \in \mathcal{L}(\vec{D}')$ as desired.

The assertion of disjointness follows directly from the fact that every function $f : [n] \to \mathbb{P}'$ has a unique linear permutation $w \in \mathcal{S}_n$ such that f is also an enriched w-partition, hence an enriched toric $[w]$-partition. Such a linear permutation w can be similarly constructed as in the proof of Lemma 3.4, so the details are omitted. This completes the proof. □

For convenience, we always assume the label set to be $[n]$. However, the definitions of (toric) DAGs and (toric) posets can be extended to any finite subset of $\mathbb{P}$ as the set of labels, with all consequent conclusions continuing to hold.

5. Weight enumerators and the algebra of cyclic peaks

In this section, we first review, in Subsection 5.1, the weight enumerators for enriched P-partitions defined by Stembridge [11], slightly extended to the context of DAGs. These enumerators span an algebra Π, the algebra of peaks, which is a graded subring of QSym. We then introduce, in Subsection 5.2, the cyclic analogue for enriched toric $[\vec{D}]$-partitions. These enumerators generate the algebra of cyclic peaks Λ, defined in Subsection 5.3, which is a graded subring of cQSym. We also introduce, in Subsection 5.4, the corresponding order polynomial and provide two proofs for an explicit formula of its evaluation on a permutation. Finally, in Subsection 5.5, we provide another characterization of the algebra of peak number $\mathcal{A}_{\mathrm{pk}}$, as opposed to the characterization given by Gessel and Zhuang [6, Theorem 4.8 (b)].

5.1. Weight enumerator for enriched $\vec{D}$-partitions. Suppose $\vec{D}$ is a DAG on $[n]$. Define the *weight enumerator for enriched $\vec{D}$-partitions* to be the formal power series

$$\Delta_{\vec{D}} := \sum_{f\in\mathcal{E}(\vec{D})} \prod_{i\in[n]} x_{|f(i)|},$$

where $\mathcal{E}(\vec{D})$ is the set of enriched $\vec{D}$-partitions. By the Fundamental Lemma 3.4, one has

$$\Delta_{\vec{D}} = \sum_{w\in\mathcal{L}(\vec{D})} \Delta_w. \tag{10}$$

It is clear from equation (8) that Δ_w is a homogeneous quasi-symmetric function. More generally, $\Delta_{\vec{D}}$ is a homogeneous quasi-symmetric function in QSym.

It also follows from equation (8) that the weight enumerator Δ_w depends only on the descent set $\operatorname{Des} w$. A less obvious but important observation, that Δ_w depends only on the peak set $\operatorname{Pk} w$, will follow directly from the following proposition, proved by Stembridge in [11].

Proposition 5.1 ([11, Proposition 2.2]). *As a quasi-symmetric function, Δ_w has the following expansion in terms of monomial quasi-symmetric functions:*

$$\Delta_w = \sum_{\substack{E\subseteq[n-1]:\\ \operatorname{Pk} w\subseteq E\cup(E+1)}} 2^{|E|+1} M_{n,E}, \tag{11}$$

where the set $E+1$ is defined by (1).

Example 5.2. In Example 4.4, we have $\mathcal{L}^{\mathrm{tor}}([\vec{D}_3]) = \{[2431],[2413]\}$. By Definition 5.4 and equation (12) in the next subsection, one has

$$\begin{aligned}\Delta^{\mathrm{cyc}}_{[\vec{D}_3]} &= \Delta^{\mathrm{cyc}}_{[2431]} + \Delta^{\mathrm{cyc}}_{[2413]}\\ &= \Delta_{2431}+\Delta_{4312}+\Delta_{3124}+\Delta_{1243}+\Delta_{2413}+\Delta_{4132}+\Delta_{1324}+\Delta_{3241}.\end{aligned}$$

Applying the proposition above, we can calculate each summand as follows:

$$\begin{aligned}\Delta_{2431} &= \sum_{E\subseteq[3]:\ \{2\}\subseteq E\cup(E+1)} 2^{|E|+1} M_{4,E}\\ &= 2^4 M_{4,\{1,2,3\}} + 2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}}) + 2^2(M_{4,\{1\}}+M_{4,\{2\}}),\end{aligned}$$

$$\begin{aligned}\Delta_{4312} &= \sum_{E\subseteq[3]:\ \varnothing\subseteq E\cup(E+1)} 2^{|E|+1}M_{4,E}\\ &= 2^4M_{4,\{1,2,3\}}+2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}})\\ &\qquad +2^2(M_{4,\{1\}}+M_{4,\{2\}}+M_{4,\{3\}})+2M_{4,\varnothing},\end{aligned}$$

$$\begin{aligned}\Delta_{3124} &= \sum_{E\subseteq[3]:\ \varnothing\subseteq E\cup(E+1)} 2^{|E|+1}M_{4,E}\\ &= 2^4M_{4,\{1,2,3\}}+2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}})\\ &\qquad +2^2(M_{4,\{1\}}+M_{4,\{2\}}+M_{4,\{3\}})+2M_{4,\varnothing},\end{aligned}$$

$$\begin{aligned}\Delta_{1243} &= \sum_{E\subseteq[3]:\ \{3\}\subseteq E\cup(E+1)} 2^{|E|+1}M_{4,E}\\ &= 2^4M_{4,\{1,2,3\}}+2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}})+2^2(M_{4,\{2\}}+M_{4,\{3\}}),\end{aligned}$$

$$\begin{aligned}\Delta_{2413} &= \sum_{E\subseteq[3]:\ \{2\}\subseteq E\cup(E+1)} 2^{|E|+1}M_{4,E}\\ &= 2^4M_{4,\{1,2,3\}}+2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}})+2^2(M_{4,\{1\}}+M_{4,\{2\}}),\end{aligned}$$

$$\begin{aligned}\Delta_{4132} &= \sum_{E\subseteq[3]:\ \{3\}\subseteq E\cup(E+1)} 2^{|E|+1}M_{4,E}\\ &= 2^4M_{4,\{1,2,3\}}+2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}})+2^2(M_{4,\{2\}}+M_{4,\{3\}}),\end{aligned}$$

$$\begin{aligned}\Delta_{1324} &= \sum_{E\subseteq[3]:\ \{2\}\subseteq E\cup(E+1)} 2^{|E|+1}M_{4,E}\\ &= 2^4M_{4,\{1,2,3\}}+2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}})+2^2(M_{4,\{1\}}+M_{4,\{2\}}),\end{aligned}$$

$$\begin{aligned}\Delta_{3241} &= \sum_{E\subseteq[3]:\ \{3\}\subseteq E\cup(E+1)} 2^{|E|+1}M_{4,E}\\ &= 2^4M_{4,\{1,2,3\}}+2^3(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}})+2^2(M_{4,\{2\}}+M_{4,\{3\}}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta^{\mathrm{cyc}}_{[\vec{D}_3]} &= 2^4\cdot 8M_{4,\{1,2,3\}}+2^3\cdot 8\left(M_{4,\{1,2\}}+M_{4,\{1,3\}}+M_{4,\{2,3\}}\right)\\ &\quad +2^2\cdot 5\left(M_{4,\{1\}}+M_{4,\{3\}}\right)+2^2\cdot 8M_{4,\{2\}}+2\cdot 2M_{4,\varnothing}\\ &= 2\cdot 2^4M^{cyc}_{(1,1,1,1)}+8\cdot 2^3M^{cyc}_{(2,1,1)}+5\cdot 2^2M^{cyc}_{(3,1)}+4\cdot 2^2M^{cyc}_{(2,2)}+2\cdot 2M^{cyc}_{(4)},\end{aligned}$$

where the second equality follows from Table 1.

As a counterpart, the weight enumerator Δ_w also has an expansion in terms of another basis: the fundamental quasi-symmetric functions.

Proposition 5.3 ([11, Proposition 3.5]). *As a quasi-symmetric function, Δ_w has the following expansion in terms of fundamental quasi-symmetric functions:*

$$\Delta_w = 2^{\mathrm{pk}\, w+1} \sum_{D\subseteq[n-1]:\ \mathrm{Pk}\, w\subseteq D\triangle(D+1)} F_{n,D}.$$

Here $\triangle$ denotes symmetric difference, that is, $D\triangle E = (D\cup E)\smallsetminus(D\cap E)$.

We say that a subset $S\subseteq[n]$ is a *peak set in* $[n]$ if $\mathrm{Pk}\, w = S$ for some $w\in\mathcal{S}_n$. For any peak set S in $[n]$, from the above proposition we can define an associated quasi-symmetric function by

$$K_{n,S} := \Delta_w,$$

where w is any permutation with peak set S. It follows that $\Delta_w = K_{n,\mathrm{Pk}\, w}$ and one can rewrite equation (10) as

$$\Delta_{\vec{D}} = \sum_{w\in\mathcal{L}(\vec{D})} K_{n,\mathrm{Pk}\, w}.$$

Let Π_n denote the space of quasi-symmetric functions spanned by $K_{n,S}$, taken over all peak sets in $[n]$, and set $\Pi = \oplus_{n\geqslant 0}\Pi_n$. In [11] Stembridge referred to Π as the algebra of peaks, and proved that Π is a graded subring of QSym.

5.2. Weight enumerator for enriched toric $[\vec{D}]$-partitions.

Definition 5.4. *For a given toric poset $[\vec{D}]$ with vertex set $[n]$, we define the* weight enumerator for enriched toric $[\vec{D}]$-partitions *by the formal power series*

$$\Delta^{\mathrm{cyc}}_{[\vec{D}]} := \sum_{f\in\mathcal{E}^{\mathrm{tor}}([\vec{D}])} \prod_{i\in[n]} x_{|f(i)|}.$$

Namely, for integer $k>0$ we assign the weight x_k to both f-values k and $-k$.

As a direct consequence of the Fundamental Lemma 4.6, one has

$$\Delta^{\mathrm{cyc}}_{[\vec{D}]} = \sum_{[w]\in\mathcal{L}^{\mathrm{tor}}([\vec{D}])} \Delta^{\mathrm{cyc}}_{[w]}. \tag{12}$$

Therefore, it suffices to discuss $\Delta^{\mathrm{cyc}}_{[w]}$ for cyclic permutations $[w]$. It follows from the formula (9) that $\Delta^{\mathrm{cyc}}_{[w]}$ can be expressed in terms of the weight enumerators $\{\Delta_v\}$ as

$$\Delta^{\mathrm{cyc}}_{[w]} = \sum_{v\in[w]} \Delta_v. \tag{13}$$

Moreover, $\Delta^{\mathrm{cyc}}_{[w]}$ has the following expression.

Proposition 5.5. *For any given cyclic permutation $[w]$ of length n, we have*

$$\Delta^{cyc}_{[w]} = \sum_{\substack{E\subseteq[n]:\\ \mathrm{cPk}(w)\subseteq E\cup(E+1)}} 2^{|E|} M^{cyc}_{n,E}, \tag{14}$$

with $E+1$ defined by (1). *The sum is independent of the choice of representative w of $[w]$.*

Proof. The independence of the choice of representatives is a result of the following two observations:

(a) If E and E' only differ by a cyclic shift, one has $|E|=|E'|$ and $M^{cyc}_{n,E} = M^{cyc}_{n,E'}$.

(b) For two representatives w and w' of $[w]$,

$$\{E \subseteq [n] : \mathrm{cPk}(w) \subseteq E \cup (E+1)\} = \{E' \subseteq [n] : \mathrm{cPk}(w') \subseteq E' \cup (E'+1)\} + i,$$

for some $i \in [n]$, namely, the two sets only differ by a cyclic shift.

Now fix a representative w of $[w]$. We rewrite both sides of equation (14) as follows:

$$\text{RHS} \overset{(i)}{=} \sum_{\substack{F \subseteq [n]: \\ \mathrm{cPk}(w) \subseteq F \cup (F+1)}} 2^{|F|} \sum_{f \in F} M_{n,(F-f) \cap [n-1]} \overset{(ii)}{=} \sum_{E \subseteq [n-1]} 2^{|E|+1} \alpha_E M_{n,E} \tag{15}$$

where $\alpha_E = \# A_E$, with

$$A_E = \{(F, f) : f \in F \subseteq [n] \text{ with } \mathrm{cPk}(w) \subseteq F \cup (F+1),\ E = (F-f) \cap [n-1]\}.$$

Here equality (i) is a result of applying equation (5) to move from cyclic monomial to monomial quasi-symmetric functions, while equality (ii) is obtained by calculating the coefficient of $M_{n,E}$. It is noted that in equality (ii), for each pair $(F, f) \in A_E$, we have $n \in F - f$. As a result, $E = (F-f) \cap [n-1] = (F-f) \smallsetminus \{n\}$. Therefore their cardinalities satisfy $|F| = |E| + 1$.

$$\begin{aligned} \text{LHS} &\overset{(i)'}{=} \sum_{v \in [w]} \Delta_v \\ &\overset{(ii)'}{=} \sum_{v \in [w]} \sum_{\substack{E \subseteq [n-1]: \\ \mathrm{Pk}(v) \subseteq E \cup (E+1)}} 2^{|E|+1} M_{n,E} \\ &\overset{(iii)'}{=} \sum_{E \subseteq [n-1]} \beta_E 2^{|E|+1} M_{n,E} \end{aligned} \tag{16}$$

where $\beta_E = \# B_E$ with

$$B_E = \{i \in [n] : (\mathrm{cPk}\, w - i) \cap [2, n-1] \subseteq E \cup (E+1)\}.$$

Equality $(i)'$ follows from equation (13). Equality $(ii)'$ is obtained by applying equation (11) to express the weight enumerator Δ_v in terms of monomial quasi-symmetric functions.

Equality $(iii)'$ follows from the observation

$$\{\{\, \mathrm{Pk}(v) : v \in [w] \,\}\} = \{\{\, (\mathrm{cPk}\, w - i) \cap [2, n-1] : i \in [n] \,\}\}.$$

By comparing equations (15) and (16), it suffices to prove that $\alpha_E = \beta_E$ for every $E \subseteq [n-1]$, or equivalently, to construct a bijection between A_E and B_E.

Set $\theta_E : A_E \to B_E$ as $\theta_E(F, f) = f$. To prove that this map is well-defined, it suffices to show that for each $(F, f) \in A_E$, we have $f \in B_E$. It follows from the definition of A_E that $F - f = E \cup \{n\}$. Applying the operation on both sides of the inclusion $\mathrm{cPk}(w) \subseteq F \cup (F+1)$, we get

$$\mathrm{cPk}(w) - f \subseteq (F-f) \cup (F-f+1) = E \cup (E+1) \cup \{1, n\}.$$

Hence $(\mathrm{cPk}\, w - f) \cap [2, n-1] \subseteq E \cup (E+1)$ and $f \in B_E$.

Conversely, define $\sigma_E : B_E \to A_E$ by $\sigma_E(f) = (\,(E+f) \cup \{f\}, f\,)$. One can similarly check that this map is well-defined.

It is straightforward to verify that θ_E and σ_E are inverse to each other, hence we get a bijection between A_E and B_E. This finishes the proof. □

Remark 5.6. As a direct corollary of the above proposition, $\Delta^{\mathrm{cyc}}_{[\vec{D}]}$ is a homogeneous cyclic quasi-symmetric function of degree n, and that $\Delta^{\mathrm{cyc}}_{[w]}$ depends only on $\mathrm{cPk}[w]$, or equivalently (by Remark 1.2) on $\mathrm{cPk}\, w$ for any representative w.

Example 5.7. Let $w = 1243$ so that $[w] = \{1243, 3124, 4312, 2431\}$ and $\mathrm{cPk}(w) = \{3\}$; $\pi = 1324$, $[\pi] = \{1324, 4132, 2413, 3241\}$ and $\mathrm{cPk}(\pi) = \{2, 4\}$.

To use Proposition 5.5 for calculation, we first need all possible choices of $E \subseteq [4]$ satisfying $\{3\} = \mathrm{cPk}\, w \subseteq E \cup (E + 1)$, which are

$$\{1,2,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2\}, \{3\},$$

with corresponding monomial cyclic quasi-symmetric functions

$$\begin{aligned} M^{cyc}_{(1,1,1,1)} &= M^{cyc}_{4,\{1,2,3,4\}}, \\ M^{cyc}_{(2,1,1)} &= M^{cyc}_{4,\{1,2,3\}} = M^{cyc}_{4,\{1,2,4\}} = M^{cyc}_{4,\{1,3,4\}} = M^{cyc}_{4,\{2,3,4\}}, \\ M^{cyc}_{(3,1)} &= M^{cyc}_{4,\{1,2\}} = M^{cyc}_{4,\{2,3\}} = M^{cyc}_{4,\{3,4\}}, \\ M^{cyc}_{(2,2)} &= M^{cyc}_{4,\{1,3\}} = M^{cyc}_{4,\{2,4\}}, \\ M^{cyc}_{(4)} &= M^{cyc}_{4,\{2\}} = M^{cyc}_{4,\{3\}}. \end{aligned}$$

Applying equation (14), we have

$$\Delta^{cyc}_{[w]} = 2^4 M^{cyc}_{(1,1,1,1)} + 4 \cdot 2^3 M^{cyc}_{(2,1,1)} + 3 \cdot 2^2 M^{cyc}_{(3,1)} + 2 \cdot 2^2 M^{cyc}_{(2,2)} + 2 \cdot 2 M^{cyc}_{(4)}.$$

Similarly for π, all possible choices of $E \subseteq [4]$ satisfying $\{2, 4\} = \mathrm{cPk}\, \pi \subseteq E \cup (E+1)$ are as follows:

$$\{1,2,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\},$$

hence by equation (14),

$$\Delta^{cyc}_{[\pi]} = 2^4 M^{cyc}_{(1,1,1,1)} + 4 \cdot 2^3 M^{cyc}_{(2,1,1)} + 2 \cdot 2^2 M^{cyc}_{(3,1)} + 2 \cdot 2^2 M^{cyc}_{(2,2)}.$$

It follows from the calculation above that

$$\Delta^{cyc}_{[w]} + \Delta^{cyc}_{[\pi]} = 2 \cdot 2^4 M^{cyc}_{(1,1,1,1)} + 8 \cdot 2^3 M^{cyc}_{(2,1,1)} + 5 \cdot 2^2 M^{cyc}_{(3,1)} + 4 \cdot 2^2 M^{cyc}_{(2,2)} + 2 \cdot 2 M^{cyc}_{(4)},$$

which is exactly what we obtained in Example 5.2.

5.3. Algebra of cyclic peaks. We say S is a *cyclic peak set in* $[n]$ if there exists some $w \in \mathcal{S}_n$ with $\mathrm{cPk}\, w = S$. It follows from Proposition 5.5 that, for any cyclic peak set S in $[n]$, we can define an associated cyclic quasi-symmetric function by

$$K^{\mathrm{cyc}}_{n,S} := \Delta^{\mathrm{cyc}}_{[w]},$$

for any permutation w with $\mathrm{cPk}\, w = S$. One can observe that $\Delta^{\mathrm{cyc}}_{[w]} = K^{\mathrm{cyc}}_{n,\mathrm{cPk}\, w}$ and rewrite equation (12) as

$$\Delta^{\mathrm{cyc}}_{[\vec{D}]} = \sum_{[w] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])} K^{\mathrm{cyc}}_{n,\mathrm{cPk}\, w}.$$

Moreover, it follows from formula (9) that $K^{\mathrm{cyc}}_{n,S}$ can be expressed in terms of the quasi-symmetric functions $\{K_{n,T}\}$ as

$$K^{\mathrm{cyc}}_{n,S} = \sum_{\sigma \in [w]} K_{n,\mathrm{Pk}\, \sigma},$$

where $\mathrm{cPk}\, w = S$.

Let Λ_n denote the vector space of cyclic quasi-symmetric functions spanned by $K^{\mathrm{cyc}}_{n,S}$ where S ranges over cyclic peak sets in $[n]$, and set $\Lambda = \oplus_{n \geqslant 0} \Lambda_n$. We will show that Λ is an algebra by proving that the product of $K^{\mathrm{cyc}}_{n,U}$ and $K^{\mathrm{cyc}}_{n,T}$ is a linear combination of $K^{\mathrm{cyc}}_{n,S}$'s. We call Λ the *algebra of cyclic peaks*.

Lemma 5.8. *The $K^{\mathrm{cyc}}_{n,S}$ are linearly independent, where S are, up to cyclic shift, distinct cyclic peak sets.*

Proof. For this proof, we totally order the subsets of $[n-1]$, first by cardinality, then by the lexicographic order. We therefore have

$$\varnothing \lhd \{1\} \lhd \{2\} \lhd \cdots \lhd \{n-1\} \lhd \{1,2\} \lhd \{1,3\} \lhd \cdots \lhd \{1,n-1\} \lhd \{2,3\} \lhd \{2,4\} \lhd \cdots.$$

One can similarly order the compositions of n as

$$(n) \lhd (1,n-1) \lhd (2,n-2) \lhd \cdots \lhd (1,1,n-2) \lhd (1,2,n-3) \lhd \cdots \lhd (1,n-2,1) \lhd \cdots.$$

Noting that $K^{\mathrm{cyc}}_{n,S} = K^{\mathrm{cyc}}_{n,S'}$ if the sets S and S' only differ by a cyclic shift, we will always assume the index set S to be the least among all its cyclic shifts. It is not hard to see that $\psi(S)$ is also the least composition among all its cyclic shifts, where ψ is defined by equation (3).

We now show that the matrix of $\{K^{\mathrm{cyc}}_{n,S}\}$ with respect to the basis $\{M^{\mathrm{cyc}}_{n,L}\}$ has full rank. Then the linear independence of $\{K^{\mathrm{cyc}}_{n,S}\}$ will follow immediately from the fact that the monomial cyclic quasi-symmetric functions form a basis of cQSym.

Let us fix n and suppose $\{S_1 \lhd S_2 \lhd \ldots \lhd S_m\}$ is the set of all distinct cyclic peak sets in $[n]$. Given a $K^{\mathrm{cyc}}_{n,S}$, suppose $|S| = k$ and $S = \{1 = s_1 < s_2 < \ldots < s_k\}$ for some $n \geqslant 2$ and $k \geqslant 1$. Here all indices are taken modulo k unless otherwise noted. To each K^{cyc}_S we associate a corresponding monomial quasi-symmetric function by $F(K^{\mathrm{cyc}}_{n,S}) = M^{\mathrm{cyc}}_{n,f(S)}$, where

$$f(S) = \{s_1, s_2 - 1, \ldots, s_k - 1\}.$$

Since S is a cyclic peak set, elements in $f(S)$ are distinct. So $f(S)$ is a set and F is well-defined. If one denotes $\psi(S)$ by $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, then

$$\psi(f(S)) = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k + 1).$$

Also notice that the assumption of S being the least among its cyclic shifts (in particular, $\alpha_1 = \min\{\alpha_i\}_{i=1}^n$) ensures that $f(S)$ is also the least among all its cyclic shifts. It follows that F is injective.

Claim: Consider the matrix of $\{K^{\mathrm{cyc}}_{n,S}\}$ with respect to the basis $\{M^{\mathrm{cyc}}_{n,E}\}$. The square submatrix with columns restricted to $\{M^{\mathrm{cyc}}_{n,f(S)}\}$ is upper triangular with nonzero diagonal entries. In particular, it is invertible.

Let $A_{i,j}$ denote the coefficient of $M^{\mathrm{cyc}}_{n,f(S_j)}$ in the expression of K^{cyc}_{n,S_i} in terms of monomial cyclic quasi-symmetric functions and set $A = (A_{i,j})$ to be the $m \times m$ matrix that we need to consider. To prove that A is an upper triangular matrix, it suffices to show that $A_{i,j} = 0$ if $i > j$, or equivalently, the term $M^{\mathrm{cyc}}_{n,f(S_j)}$ does not appear in the expression of K^{cyc}_{n,S_i} if $S_j \lhd S_i$.

Suppose towards a contradiction that $A_{i,j} \neq 0$ for some $i > j$. It then follows from Proposition 5.5 that there exists some $E \subseteq [n]$ such that

$$S_i \subseteq E \cup (E+1) \quad \text{and} \quad M^{\mathrm{cyc}}_{n,f(S_j)} = M^{\mathrm{cyc}}_{n,E}.$$

Since S_i is a cyclic peak set, e and $e+1$ cannot be in S_i at the same time. It then follows from the assumption $S_i \subseteq E \cup (E+1)$ that $|E| \geqslant |S_i|$. The condition $i > j$ implies $S_i \rhd S_j$, hence $|S_i| \geqslant |S_j| = |f(S_j)| = |E|$. Therefore, we only need to consider the cases when S_i and S_j have the same cardinality.

Suppose

$$\psi(S_i) = (\alpha_1, \alpha_2, \ldots, \alpha_k), \quad \psi(S_j) = (\beta_1, \beta_2, \ldots, \beta_k), \quad \psi(E) = (\beta'_1, \beta'_2, \ldots, \beta'_k).$$

Then $\psi(E) = (\beta'_1, \beta'_2, \ldots, \beta'_k)$ is a cyclic shift of

$$\psi(f(S_j)) = (\beta_1 - 1, \beta_2, \ldots, \beta_{k-1}, \beta_k + 1).$$

Assume

$$(\beta'_q, \beta'_{q+1}, \dots, \beta'_{q-1}) = (\beta_1 - 1, \beta_2, \dots, \beta_{k-1}, \beta_k + 1),$$

for some $q \in [k]$. Since $S_i \subseteq E \cup (E+1)$, $|S_i| = |E|$, and S_i does not have cyclically consecutive elements, each $a \in S_i$ corresponds to a unique $a' \in E$ where a' equals either a or $a-1$ (considered modulo n). A crucial observation therefore follows:

$$|(\alpha_r + \dots + \alpha_s) - (\beta'_r + \dots + \beta'_s)| \leqslant 1 \quad \text{for } r, s \in [k]. \tag{17}$$

In particular,

$$\alpha_1 \leqslant \alpha_q \leqslant \beta'_q + 1 = (\beta_1 - 1) + 1 = \beta_1,$$

where the first inequality comes from the fact that S_i is the least among its cyclic shifts, and the second one follows from (17) by taking $r = s = q$. Note that $S_j \lhd S_i$ implies $\psi(S_j) \lhd \psi(S_i)$, so $\beta_1 \leqslant \alpha_1$. Thus, necessarily,

$$\alpha_1 = \alpha_q = \beta'_q + 1 = \beta_1.$$

It then follows from the inequality (17) that for any $r \in [k-1]$,

$$\alpha_q + \dots + \alpha_{q+r} \leqslant \beta'_q + \dots + \beta'_{q+r} + 1 = \beta_1 + \dots + \beta_{1+r} + \delta_{1+r,k},$$

where $\delta_{1+r,k} = 1$ if $1 + r = k$ and 0 otherwise.

If $\alpha_{q+r} = \beta_{1+r}$ for every $r \in [k-2]$, then $\psi(S_i)$ is a cyclic shift of $\psi(S_j)$, which contradicts our assumption $S_i \rhd S_j$. Now assume that $t \in [k-2]$ is the smallest index such that $\alpha_{q+t} \neq \beta_{1+t}$. Then necessarily $\alpha_{q+t} < \beta_{1+t}$, and

$$(\alpha_q, \dots, \alpha_{q+t}) \lhd (\beta_1, \dots, \beta_{1+t}).$$

Combined with the inequality $(\alpha_1, \dots, \alpha_{1+t}) \unlhd (\alpha_q, \dots, \alpha_{q+t})$, as $(\alpha_1, \dots, \alpha_k)$ is the least among all its cyclic shifts, one has

$$(\alpha_1, \dots, \alpha_{1+t}) \lhd (\beta_1, \dots, \beta_{1+t}).$$

This implies that $\psi(S_i) \lhd \psi(S_j)$, which is a contradiction to the assumption $S_i \rhd S_j$. Moreover, it is not hard to see that $A_{i,i} \neq 0$ as $S_i \subseteq f(S_i) \cup (f(S_i) + 1)$, therefore the diagonal entries are nonzero. This proves the claim, hence the lemma. ☐

Define *the union of digraphs* $\vec{D} \uplus \vec{E}$ to be the digraph with vertices $V(\vec{D}) \cup V(\vec{E})$ and arcs $A(\vec{D}) \cup A(\vec{E})$. The next result follows easily from the definition.

Proposition 5.9. *If $\vec{D}$ and $\vec{E}$ are two DAGs on disjoint subsets of* $\mathbb{P}$, *then*

$$\Delta^{\mathrm{cyc}}_{[\vec{D} \uplus \vec{E}]} = \Delta^{\mathrm{cyc}}_{[\vec{D}]} \cdot \Delta^{\mathrm{cyc}}_{[\vec{E}]}. \tag{18}$$

The proposition above yields a combinatorial proof that Λ is an algebra.

Proposition 5.10. Λ *is a graded subring of* cQSym.

Proof. As a subspace of cQSym, Λ naturally inherits the addition and multiplication operations from cQSym.

To show that Λ is closed under multiplication, let $K^{\mathrm{cyc}}_{n,U} \in \Lambda_m$ and $K^{\mathrm{cyc}}_{n,T} \in \Lambda_n$, where U and T are cyclic peak sets in $[m]$ and $[n]$ respectively. That is to say, there exist $\pi \in \mathcal{S}_m$ and $w \in \mathcal{S}_n$ such that $\mathrm{cPk}\,\pi = U$, $\mathrm{cPk}\,w = T$. For the purpose of constructing two corresponding disjoint DAGs, we standardize $w = w_1 w_2 \dots w_n \in \mathcal{S}_n$ to $\{m+1, m+2, \dots, m+n\}$, that is, construct $w' = w'_1 w'_2 \dots w'_n$ where $w'_i = w_i + m$ for $i \in [n]$. As a consequence of equation (18), we have

$$K^{\mathrm{cyc}}_{n,U} \cdot K^{\mathrm{cyc}}_{n,T} = \Delta^{\mathrm{cyc}}_{[\pi]} \cdot \Delta^{\mathrm{cyc}}_{[w']} = \Delta^{\mathrm{cyc}}_{[\pi \uplus w']} = \sum_{\sigma \in \mathcal{L}^{\mathrm{tor}}([\pi \uplus w'])} K^{\mathrm{cyc}}_{n,\mathrm{cPk}\,\sigma}. \tag{19}$$

This completes the proof. ☐

In terms of another basis of cQSym, the fundamental cyclic quasi-symmetric functions, $K^{\mathrm{cyc}}_{n,S}$ has the following expansion.

Proposition 5.11. *For any cyclic peak set S in $[n]$, we have*

$$2^{-|S|}K^{\mathrm{cyc}}_{n,S} = \sum_{\substack{E\subseteq[n]:\\ S\subseteq E\triangle(E+1)}} F^{cyc}_{n,E},$$

where $\triangle$ denotes symmetric difference.

Proof. Since $F^{cyc}_{n,E} = \sum_{F\supseteq E} M^{cyc}_{n,F}$, the coefficient of $M^{cyc}_{n,F}$ on the right-hand side of the above expansion is $|\{E \subseteq F : S \subseteq E\triangle(E+1)\}|$.

To count this set, we need the following observations. For each $k \in S \subseteq E\triangle(E+1)$, exactly one of $k-1$ and k is in E. It follows from $E \subseteq F$ that at least one of $k-1$ and k is in F. So one has the following two cases:

(1) If both $k, k-1 \in F$, then E must contain exactly one of k or $k-1$.
(2) If only one of $k, k-1 \in F$, then E must contain this element.

Note that the restrictions above only involve two adjacent numbers.

Also notice that if $k \in F$ but neither k nor $k+1$ is in S, then k is free to be in E or not. Denote

$$S_1 = \{k \in S \mid \text{both } k, k-1 \text{ are in } F\},$$
$$S_2 = \{k \in S \mid k \in E, k-1 \notin F\},$$
$$S_3 = \{k \in S \mid k \notin E, k-1 \in F\}.$$

Then we have a set partition $S = S_1 \cup S_2 \cup S_3$. Therefore if we denote $s_i = \#S_i$ for $i \in \{1, 2, 3\}$, we have $|S| = s_1 + s_2 + s_3$.

Denote now

$$T = \{k \in F \mid \text{none of } k, k+1 \text{ is in } S\}.$$

By the definition of a peak set, numbers in S are not adjacent. Hence we have the following partition of F into disjoint sets:

$$F = S_1 \cup (S_1 - 1) \cup S_2 \cup (S_3 - 1) \cup T.$$

It follows that $|F| = 2s_1 + s_2 + s_3 + t$ with $t = |T|$. Hence, the number of choices for E is

$$2^{s_1+t} = 2^{s_1+|F|-2s_1-s_2-s_3} = 2^{|F|-(s_1+s_2+s_3)} = 2^{|F|-|S|}.$$

So we have

$$\sum_{\substack{E\subseteq[n]:\\ S\subseteq E\triangle(E+1)}} F^{cyc}_{n,E} = \sum_{\substack{F\subseteq[n]:\\ S\subseteq F\cup(F+1)}} 2^{|F|-|S|}M^{cyc}_{n,F}.$$

By equation (14), this quantity is $2^{-|S|}K^{cyc}_{n,S}$. □

5.4. Order polynomials $\Omega^{\mathrm{cyc}}([\vec{D}], m)$. Given a DAG $\vec{D}$, we can define the *order polynomial of enriched $\vec{D}$-partitions*, $\Omega(\vec{D}, m)$, by

$$\Omega(\vec{D}, m) = \Delta_{\vec{D}}(1^m),$$

where $\Delta_{\vec{D}}(1^m)$ means that we set $x_1 = \cdots = x_m = 1$, and $x_k = 0$ for $k > m$. In fact, $\Omega(\vec{D}, m)$ counts the number of enriched $\vec{D}$-partitions with absolute value at most m.

Stembridge computed the corresponding generating function as follows:

Theorem 5.12 ([11, Theorem 4.1]). *For a given $w \in \mathcal{S}_n$, one has*

$$\sum_m \Omega(w, m)t^m = \frac{1}{2}\left(\frac{1+t}{1-t}\right)^{n+1}\left(\frac{4t}{(1+t)^2}\right)^{1+\operatorname{pk} w}. \tag{20}$$

It is not hard to see that $\Omega(\vec{D},t)$ is indeed a polynomial in t. If $\vec{D}$ has vertex set V and $|V|=n$, then

$$\Omega(\vec{D},m)=\sum_{k=1}^{n} c_k\binom{m}{k},$$

where c_k denotes the number of $f\in\mathcal{E}(\vec{D})$ such that $\{|f(x)|\colon x\in V\}=[k]$. For any fixed k, $\binom{m}{k}$ is a polynomial in m of degree k. Since c_k and n are constants, it follows that the summation is also a polynomial in m. This verifies that $\Omega([\vec{D}],t)$ is a polynomial.

Similarly, we define the *order polynomial of enriched toric* $[\vec{D}]$*-partitions*, $\Omega^{\mathrm{cyc}}([\vec{D}],m)$, by

$$\Omega^{\mathrm{cyc}}([\vec{D}],m)=\Delta^{\mathrm{cyc}}_{[\vec{D}]}(1^m).$$

The following result is the toric analogue of formula (20).

Proposition 5.13. *Given $w\in\mathcal{S}_n$, then*

$$\sum_m \Omega^{\mathrm{cyc}}([w],m)t^m=\left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk} w}\left(\frac{1+t}{1-t}\right)^{n-1}\left(\operatorname{cpk} w+\frac{2nt}{(1-t)^2}\right). \tag{21}$$

This right side of the equation does not depend on the choice of representative w, as they all have the same cyclic peak number.

Proof. By the definition of order polynomial,

$$\begin{aligned}
\sum_m \Omega^{\mathrm{cyc}}([w],m)t^m &= \sum_m \Delta^{\mathrm{cyc}}_{[w]}(1^m)t^m\\
&=\sum_m\sum_{v\in[w]}\Delta_v(1^m)t^m\\
&=\sum_{v\in[w]}\sum_m\Delta_v(1^m)t^m\\
&=\sum_{v\in[w]}\sum_m\Omega(v,m)t^m\\
&=\sum_{v\in[w]}\frac{1}{2}\left(\frac{1+t}{1-t}\right)^{n+1}\left(\frac{4t}{(1+t)^2}\right)^{1+\operatorname{pk} v},
\end{aligned}$$

where the last equality is obtained by applying (20).

Observe that each representative of $[w]$ will either start with a cyclic peak, end with a cyclic peak, or none of the two ends are cyclic peaks, which will yield peak number $\operatorname{cpk} w-1$, $\operatorname{cpk} w-1$ or $\operatorname{cpk} w$ respectively. The number of those representatives with a cyclic peak at one end is $2\operatorname{cpk} w$. It follows that

$$\begin{aligned}
\sum_m \Omega^{\mathrm{cyc}}([w],m)t^m &= \frac{2\operatorname{cpk} w}{2}\left(\frac{1+t}{1-t}\right)^{n+1}\left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk} w}\\
&\qquad+\frac{n-2\operatorname{cpk} w}{2}\left(\frac{1+t}{1-t}\right)^{n+1}\left(\frac{4t}{(1+t)^2}\right)^{1+\operatorname{cpk} w}\\
&=\left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk} w}\left(\frac{1+t}{1-t}\right)^{n-1}\left(\operatorname{cpk} w+\frac{2nt}{(1-t)^2}\right).
\end{aligned}$$

This completes the proof. □

Notice that by taking the coefficient of t^m on both sides of equation (21), one can get an expression for the order polynomial of enriched $\vec{D}$-partitions $\Omega^{\text{cyc}}([w],m)$ in an algebraic manner. It would be desirable to derive $\Omega^{\text{cyc}}([w],m)$ combinatorially. For this purpose, we first give a combinatorial proof of a formula for $\Omega(w,m)$. We define the (w,m)-marking for a permutation w and a positive integer m, and show that each (w,m)-marking corresponds to exactly $2^{2\operatorname{pk}+1}$ enriched w-partitions with absolute value at most m.

Suppose $w \in \mathcal{S}_n$, $m \in \mathbb{P}$. One can naturally extend $w = w_1 \dots w_n$ to $w' = w_0 w_1 \dots w_n w_{n+1}$ where $w_0 = w_{n+1} = \infty$. Let R_1 be the longest decreasing initial factor of w'. Now let v denote w' with R_1 deleted, and let R_2 be the longest increasing initial factor of v. Continue in this way, alternating between decreasing and increasing factors to get a factorization of w'. We call the factors *runs* and the corresponding indices *run indices*. We denote by I_j the set of run indices corresponding to a factor set R_j. Note that any extension w' will start with a decreasing run and end with an increasing run, which implies that the number of runs is always even.

Take

$$w = \begin{matrix} 1\,2\,3\,4\,5\,6 \\ \mathbf{1}\,4\,\mathbf{3}\,\mathbf{2}\,5\,6 \end{matrix}$$

as an example. The corresponding natural extension

$$w' = \begin{matrix} 0\ \ 1\,2\,3\,4\,5\,6\ \ 7 \\ \infty\,\mathbf{1}\,4\,\mathbf{3}\,\mathbf{2}\,5\,6\,\infty \end{matrix}$$

has four runs

$$R_1 = \infty 1,\ R_2 = 4,\ R_3 = 32,\ R_4 = 56\infty,$$

where the decreasing runs are in bold, and the corresponding set of run indices are

$$I_1 = \{0,1\},\ I_2 = \{2\},\ I_3 = \{3,4\},\ I_4 = \{5,6,7\}.$$

Suppose that the permutation w' has r runs. We have the following observations:

(1) The parity of i indicates the type of the run R_i. If i is even, then R_i is increasing. If i is odd, then R_i is decreasing.

(2) The total number of runs r is closely related to the peak number $\operatorname{pk} w$:

$$r = 2\operatorname{pk} w + 2.$$

We now decorate permutations with bars and marks. Bars can be inserted between adjacent columns in the two-line notation (including the space before the first column and the space after the last), whereas a column of w with index $i \in [n]$ can be marked if and only if $i, i+1 \in I_j$ for some j; in other words, if i and $i+1$ are in the same run index set. There can be multiple bars between two adjacent columns and we count them with multiplicity, while each column can be marked at most once. We will denote by $\mathcal{M}_w$ the set of indices of the columns that can be marked,

$$\mathcal{M}_w := \{i \in [n] \mid\ i, i+1 \in I_j \text{ for some } j\}.$$

We note that for a given w, the cardinality of the complement of the set $\mathcal{M}_w$ in $[n]$ is $2\operatorname{pk} w + 1$. Equivalently, one has $|\mathcal{M}_w| = n - 2\operatorname{pk} w - 1$.

Definition 5.14 ((w,m)-marking). *Suppose that w is a linear permutation and m is a positive integer. A* (w,m)-marking *is a marking of w using b bars and d marked columns, satisfying that $b + d = m - 1 - \operatorname{pk} w$.*

Example 5.15. If we set $w = 143256$ as before, and take $m = 5$, then a $(w,5)$-marking has b bars and d marked columns, such that $b + d = 3$. Two $(w,5)$-markings are provided as follows. Both have two bars and one marked column, where the marked column is in blue.

$$\begin{array}{cc} 1\ 2 \| 3\ 4\ \mathbf{5}\ 6 & \quad 1\ 2 | \mathbf{3}\ 4\ 5\ 6 | \\ 1\ 4 \| 3\ 2\ \mathbf{5}\ 6 & \quad 1\ 4 | \mathbf{3}\ 2\ 5\ 6 | \end{array}$$

Proposition 5.16. *For a given $w \in \mathcal{S}_n$, one has*

$$\Omega(w,m) = 2^{2\operatorname{pk} w+1} \sum_{k=0}^{m-1-\operatorname{pk} w} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n-2\operatorname{pk} w-1}{m-1-\operatorname{pk} w-k}, \tag{22}$$

where $\left(\!\binom{n+1}{k}\!\right)$ denotes the number of multisets on $[n+1]$ with cardinality k.

Proof. It is clear by definition that the number of possible choices for (w,m)-markings is

$$\sum_{b+d=m-1-\operatorname{pk} w} \left(\!\!\binom{n+1}{b}\!\!\right) \binom{n-2\operatorname{pk} w-1}{d} = \sum_{k=0}^{m-1-\operatorname{pk} w} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n-2\operatorname{pk} w-1}{m-1-\operatorname{pk} w-k}.$$

Therefore, it suffices to construct a $2^{2\operatorname{pk} w+1}$-to-one map from the set of enriched w-partitions with absolute value at most m to the set of all (w,m)-markings.

Given an enriched w-partition f with absolute value at most m, we can inductively associate to it a unique (w,m)-marking as follows:

We first determine, for each $k \in \mathcal{M}_w$, whether column k gets marked. Suppose $k \in I_i$ for some $i \in [r]$. We mark column k if and only if $\delta_k + \gamma_k = 1$ where

$$\delta_k := \delta\,(i \text{ is even and } f(w_k) < 0), \qquad \gamma_k := \delta\,(i \text{ is odd and } f(w_k) > 0).$$

Here the *Kronecker function* on a statement R is defined by

$$\delta(R) = \begin{cases} 1 & \text{if } R \text{ is true,} \\ 0 & \text{if } R \text{ is false.} \end{cases}.$$

As for the placement of bars, we start by putting $|f(w_1)| - 1$ bars before the first column. Inductively, suppose that $k \in I_i$ for some $k \in [2,n]$, $i \in [r]$ and we have already constructed the markings and bars on and before the $(k-1)$st column. Then the number of marks and bars strictly before the kth column is constructed to be

$$|f(w_k)| - \lceil i/2 \rceil - \delta_k. \tag{23}$$

Finally we add bars after the last column so that the total number of marks and bars is $m - \operatorname{pk} w - 1$. In this manner, we can inductively define a unique (w,m)-marking for f.

We must verify that the constructed marking is indeed a (w,m)-marking. Firstly we will verify that (23) is a weakly increasing function of k, and strictly increasing from the kth to the $(k+1)$st term if column k is marked. In other words, if $k \in I_i$ and $k+1 \in I_j$, it suffices to show that

$$|f(w_k)| - \lceil i/2 \rceil - \delta_k \leqslant |f(w_{k+1})| - \lceil j/2 \rceil - \delta_{k+1},$$

for all k, and that

$$|f(w_k)| - \lceil i/2 \rceil - \delta_k < |f(w_{k+1})| - \lceil j/2 \rceil - \delta_{k+1},$$

if column k is marked. Notice that

$$(\delta_k + \gamma_k) \cdot \delta(k \in \mathcal{M}_w) = 1$$

if column k is marked and 0 otherwise. Hence it suffices to show that

$$|f(w_k)| - \lceil i/2 \rceil - \delta_k + (\delta_k + \gamma_k) \cdot \delta(k \in \mathcal{M}_w) \leqslant |f(w_{k+1})| - \lceil j/2 \rceil - \delta_{k+1}. \tag{24}$$

By the definition of enriched P-partitions, we have

$$|f(w_k)| \leqslant |f(w_{k+1})|. \tag{25}$$

Let us consider the following cases:

(a) If $j = i$, it follows that $k \in \mathcal{M}_w$. Hence inequality (24) simplifies to

$$|f(w_k)| + \gamma_k \leqslant |f(w_{k+1})| - \delta_{k+1}.$$

If i is even, then $\gamma_k = 0$ and $w_k < w_{k+1}$. By inequality (25), one only needs to consider whether the inequality holds when $\delta_{k+1} = 1$, which implies $f(w_{k+1}) < 0$. It follows from the definition of enriched P-partitions that $f(w_k) \preceq f(w_{k+1})$, namely $|f(w_k)| < |f(w_{k+1})|$ or $f(w_k) = f(w_{k+1}) < 0$, but the second possibility contradicts $w_k < w_{k+1}$. This proves (24) in this case. The proof is similar when i is odd.

(b) If $j = i + 1$, then $k \notin \mathcal{M}_w$. Inequality (24) reduces to

$$|f(w_k)| - \lceil i/2 \rceil - \delta_k \leqslant |f(w_{k+1})| - \lceil (i+1)/2 \rceil - \delta_{k+1}.$$

If i is even, then $w_k > w_{k+1}$, $\lceil i/2 \rceil + 1 = \lceil (i+1)/2 \rceil$ and j is odd, hence $\delta_{k+1} = 0$. Therefore one only needs to prove

$$|f(w_k)| - \delta_k \leqslant |f(w_{k+1})| - 1.$$

The inequality clearly holds when $\delta_k = 1$, so it suffices to consider the case when $\delta_k = 0$. In this case, $f(w_k) > 0$, hence by the definition of enriched P-partitions, or equivalently $|f(w_{k+1})| \geqslant |f(w_k)| + 1$, proves (24). The case when i is odd is similar and left to the reader.

Secondly, one also needs to check that it is possible to add bars (possibly 0) after the last column so that the total number of marks and bars is $m - \operatorname{pk} w - 1$. In other words, the number of bars we add after the nth column is nonnegative. Notice that $\lceil \frac{i}{2} \rceil - 1$ counts the number of peaks before the k-th column. Since $n \in I_r$, this implies that $\lceil \frac{r}{2} \rceil = \operatorname{pk} w + 1$. By (23) the total number of bars and marked columns before the nth column is $|f(w_n)| - \operatorname{pk} w - 1 - \delta_n$. Together with the fact that the nth column is marked if and only if $\delta_n + \gamma_n = 1$ and $n \in \mathcal{M}_w$, the total number of bars that should be added after the nth column is

$$\begin{aligned} &m - \operatorname{pk} w - 1 - (|f(w_n)| - \operatorname{pk} w - 1 - \delta_n) - (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_w) \\ =\, &m - |f(w_n)| + \delta_n - (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_w). \end{aligned}$$

Hence the nonnegativity condition becomes

$$m - |f(w_n)| + \delta_n - (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_w) \geqslant 0,$$

or equivalently,

$$m - |f(w_n)| \geqslant (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_w) - \delta_n. \tag{26}$$

By assumption f has absolute value at most m, hence $m - |f(w_n)| \geqslant 0$. Therefore, one only needs to check that the inequality holds when $(\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_w) - \delta_n = 1$, namely $\delta_n + \gamma_n = 1$, $\delta(n \in \mathcal{M}_w) = 1$ and $\delta_n = 0$, or equivalently, $n \in I_i$ where i is odd, $f(w_n) > 0$ and $n \in \mathcal{M}_w$. This is a contradiction since $n + 1$ must be in a run where the index is even by definition, which implies that n and $n + 1$ cannot be in the same run index set, hence $n \notin \mathcal{M}_w$. This completes the proof of inequality (26). It therefore follows that the marking we constructed is indeed a (w, m)-marking.

Now we need to show that for a given (w, m)-marking, there are $2^{2 \operatorname{pk} w + 1}$ different associated functions. From the above construction we notice that for each $k \in \mathcal{M}_w$, $f(w_k)$ is uniquely determined. More precisely, whether column k gets marked determines δ_k and γ_k as k determines the parity of i, hence the sign of $f(w_k)$ is determined by the definitions of δ_k and γ_k. Suppose $k \in I_i$. The number of marks and bars strictly before the kth column determines the value $|f(w_k)| - \lceil i/2 \rceil - \delta_k$, therefore it determines $f(w_k)$ as well. The only ambiguity is about $f(w_k)$ for $k \notin \mathcal{M}_w$. As the number of marks and bars strictly before the kth column fixes the value $L = |f(w_k)| - \lceil i/2 \rceil - \delta_k$, there are two possible choices of the value $f(w_k)$ for each $k \notin \mathcal{M}_w$: if i is even, then either $f(w_k) = -(L + \lceil i/2 \rceil + 1)$ or $f(w_k) = L + \lceil i/2 \rceil$; if i is odd, then either $f(w_k) = L + \lceil i/2 \rceil$ or $f(w_k) = -(L + \lceil i/2 \rceil)$. It follows that there are $2^{2\,\mathrm{pk}\,w+1}$ different functions corresponding to a given (w, m)-marking. □

Example 5.17. Consider again the permutation $w = 143256$, and an enriched w-partition f defined as follows:

$$f(1) = 1,\ f(4) = -2,\ f(3) = -4,\ f(2) = -4,\ f(5) = -5,\ f(6) = 5.$$

The corresponding marking is the first one in Example 5.15.

We now give a combinatorial derivation of the order polynomial $\Omega^{\mathrm{cyc}}([w], m)$ by using Proposition 5.16, which was also proved combinatorially.

Corollary 5.18. *For a given $[w] \in [\mathcal{S}_n]$, one has*

$$\Omega^{\mathrm{cyc}}([w], m) = (n - 2\,\mathrm{cpk}\,w) \cdot 2^{2\,\mathrm{cpk}\,w+1} \sum_{k=0}^{m-1-\mathrm{cpk}\,w} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\,\mathrm{cpk}\,w - 1}{m - 1 - \mathrm{cpk}\,w - k}$$
$$+ \mathrm{cpk}\,w \cdot 2^{2\,\mathrm{cpk}\,w} \sum_{k=0}^{m-\mathrm{cpk}\,w} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\,\mathrm{cpk}\,w + 1}{m - \mathrm{cpk}\,w - k}.$$

This right side of the equation does not depend on the choice of representative w, as they all have the same cyclic peak number.

Proof. Notice that any representative w' of $[w]$ satisfies $\mathrm{pk}\,w' = \mathrm{cpk}[w] - 1$ if w' starts or ends with a cyclic peak, and $\mathrm{pk}\,w' = \mathrm{cpk}[w]$ otherwise. Among the n representatives of $[w]$, there are $2\,\mathrm{cpk}[w]$ with a cyclic peak at one end. Therefore, applying equation (13) we have

$$\Omega^{\mathrm{cyc}}([w], m) = \Delta^{\mathrm{cyc}}_{[w]}(1^m) = \sum_{v \in [w]} \Delta_v(1^m) = \sum_{v \in [w]} \Omega(v, m),$$

and by the previous observation and Proposition 5.16, one has

$$\begin{aligned}
\sum_{v \in [w]} \Omega(v, m) &= \sum_{v \in [w]} 2^{2\,\mathrm{pk}\,v+1} \sum_{k=0}^{m-1-\mathrm{pk}\,v} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\,\mathrm{pk}\,v - 1}{m - 1 - \mathrm{pk}\,v - k} \\
&= (n - 2\,\mathrm{cpk}[w]) \cdot 2^{2\,\mathrm{cpk}[w]+1} \sum_{k=0}^{m-1-\mathrm{cpk}[w]} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\,\mathrm{cpk}[w] - 1}{m - 1 - \mathrm{cpk}[w] - k} \\
&+ 2\,\mathrm{cpk}[w] \cdot 2^{2(\mathrm{cpk}[w]-1)+1} \sum_{k=0}^{m-1-(\mathrm{cpk}[w]-1)} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2(\mathrm{cpk}[w]-1) - 1}{m - 1 - (\mathrm{cpk}[w]-1) - k} \\
&= (n - 2\,\mathrm{cpk}\,w) \cdot 2^{2\,\mathrm{cpk}\,w+1} \sum_{k=0}^{m-1-\mathrm{cpk}\,w} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\,\mathrm{cpk}\,w - 1}{m - 1 - \mathrm{cpk}\,w - k} \\
&+ \mathrm{cpk}\,w \cdot 2^{2\,\mathrm{cpk}\,w} \sum_{k=0}^{m-\mathrm{cpk}\,w} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\,\mathrm{cpk}\,w + 1}{m - \mathrm{cpk}\,w - k}
\end{aligned}$$

The conclusion follows immediately. □

We note that the generating function of order polynomials in Proposition 5.13 can also be deduced from the corollary above. Explicitly,

$$\begin{aligned}\sum_m \Omega^{\mathrm{cyc}}([w],m)t^m &= (n-2\operatorname{cpk} w)\cdot 2^{2\operatorname{cpk} w+1}\,\frac{(1+t)^{n-2\operatorname{cpk} w-1}}{(1-t)^{n+1}}\,t^{\operatorname{cpk} w+1}\\ &\quad+\operatorname{cpk} w\cdot 2^{2\operatorname{cpk} w}\,\frac{(1+t)^{n-2\operatorname{cpk} w+1}}{(1-t)^{n+1}}\,t^{\operatorname{cpk} w}\\ &=\left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk} w}\left(\frac{1+t}{1-t}\right)^{n+1}\left((n-2\operatorname{cpk} w)\cdot\frac{2t}{(1+t)^2}+\operatorname{cpk} w\right)\\ &=\left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk} w}\left(\frac{1+t}{1-t}\right)^{n-1}\left(\operatorname{cpk} w+\frac{2nt}{(1-t)^2}\right).\end{aligned}$$

5.5. The shuffle algebra of the peak number. In [6], Gessel and Zhuang discussed shuffle-compatible permutation statistics in terms of shuffle algebras. In Theorem 4.8, they proved that the peak number pk is shuffle compatible and also characterized its shuffle algebra $\mathcal{A}_{\mathrm{pk}}$. It turns out that Proposition 5.16 gives another characterization of $\mathcal{A}_{\mathrm{pk}}$. Before we state and prove the theorem, we review some definitions and results from [6].

Suppose π and σ are two disjoint permutations of lengths m and n respectively. Then the *shuffle set* of π and σ is

$$\pi \sqcup\!\sqcup\, \sigma = \{\tau : |\tau| = m+n,\ \pi \text{ and } \sigma \text{ are subsequences of } \tau\}.$$

A statistic st is (*linear*) *shuffle compatible* if for disjoint permutations π and σ, the multiset $\{\{\,\mathrm{st}(\tau) : \tau\in\pi\sqcup\!\sqcup\,\sigma\,\}\}$ only depends on $\mathrm{st}(\pi)$, $\mathrm{st}(\sigma)$, $|\pi|$ and $|\sigma|$.

Every linear permutation statistic st induces an equivalence relation on permutations. More precisely, two permutations π and σ are st-equivalent if $\mathrm{st}(\pi)=\mathrm{st}(\sigma)$ and $|\pi|=|\sigma|$, and the st-*equivalence class of* π is denoted by $[\pi]_{\mathrm{st}}$. Moreover, if st is shuffle compatible, one can associate to st a $\mathbb{Q}$-algebra as follows: first we associate to st a $\mathbb{Q}$-vector space by taking the st-equivalence classes as a basis, then define multiplication by

$$[\pi]_{\mathrm{st}}[\sigma]_{\mathrm{st}} = \sum_{\tau\in\pi\sqcup\!\sqcup\,\sigma}[\tau]_{\mathrm{st}}.$$

Here the shuffle-compatibility of st guarantees that the above multiplication is well-defined. In this case, we call the resulting algebra the *shuffle algebra of* st and denote it by $\mathcal{A}_{\mathrm{st}}$.

We recall the characterization of the peak shuffle algebra $\mathcal{A}_{\mathrm{pk}}$ from [6]:

Theorem 5.19 ([6, Theorem 4.8 (b)]). *The linear map on* $\mathcal{A}_{\mathrm{pk}}$ *defined by*

$$[\pi]_{\mathrm{pk}}\mapsto\begin{cases}\dfrac{2^{2\operatorname{pk}\pi+1}t^{\operatorname{pk}\pi+1}(1+t)^{|\pi|-2\operatorname{pk}\pi-1}}{(1-t)^{|\pi|+1}}x^{|\pi|}, & \text{if } |\pi|\geqslant 1;\\ \dfrac{1}{1-t}, & \text{if } |\pi|=0,\end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{pk}}$ *to the span of*

$$\left\{\frac{1}{1-t}\right\}\bigcup\left\{\frac{2^{2j+1}t^{j+1}(1+t)^{n-2j-1}}{(1-t)^{n+1}}x^n\right\}_{n\geqslant 1,\,0\leqslant j\leqslant\lfloor\frac{n-1}{2}\rfloor}$$

a subalgebra of $\mathbb{Q}[[t*]][x]$*, where multiplication of formal power series in* t *is by Hadamard product.*

Now we give another characterization of $\mathcal{A}_{\mathrm{pk}}$ which follows from our Proposition 5.16.

Theorem 5.20. *The linear map on $\mathcal{A}_{\mathrm{pk}}$ defined by*

$$[\pi]_{\mathrm{pk}} \mapsto 2^{2\,\mathrm{pk}\,\pi+1} \sum_{k=0}^{m-1-\mathrm{pk}\,\pi} \left(\!\!\binom{|\pi|+1}{k}\!\!\right) \binom{|\pi|-2\,\mathrm{pk}\,\pi-1}{m-1-\mathrm{pk}\,\pi-k} x^{|\pi|}$$

is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}_{\mathrm{pk}}$ to the span of

$$\{1\} \bigcup \left\{ 2^{2j+1} \sum_{k=0}^{m-1-j} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n-2j-1}{m-1-j-k} x^n \right\}_{n\geqslant 1,\, 0\leqslant j\leqslant \lfloor \frac{n-1}{2} \rfloor}$$

a subalgebra of $\mathbb{Q}[x]^{\mathbb{N}}$, the algebra of functions $\mathbb{N} \to \mathbb{Q}[x]$ in the non-negative integer value m, with pointwise addition and multiplication.

Proof. Define a map $\kappa_m : \mathrm{QSym} \to \mathbb{Q}[x]$ by

$$\kappa_m(F_{n,L}) = K_{n,\mathrm{Pk}(L)}(1^m) x^n,$$

linearly extended to all of QSym. The following equation, [11, Equation (3.1)],

$$K_{n,\mathrm{Pk}\,\pi} \cdot K_{n,\mathrm{Pk}\,\sigma} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} K_{n,\mathrm{Pk}\,\tau},$$

implies that κ_m is a $\mathbb{Q}$-algebra homomorphism. The map that takes $F_{n,L}$ to the function $\theta_{n,L} : m \mapsto \kappa_m(F_{n,L})$ is therefore a homomorphism from QSym to $\mathbb{Q}[x]^{\mathbb{N}}$. It follows from Proposition 5.16 that

$$\kappa_m(F_{n,L}) = 2^{2\,\mathrm{pk}(L)+1} \sum_{k=0}^{m-1-\mathrm{pk}(L)} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n-2\,\mathrm{pk}(L)-1}{m-1-\mathrm{pk}(L)-k} x^n.$$

Moreover, from Theorem 5.12 one has

$$\sum_{m=0}^{\infty} 2^{2\,\mathrm{pk}(L)+1} \sum_{k=0}^{m-1-\mathrm{pk}(L)} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n-2\,\mathrm{pk}(L)-1}{m-1-\mathrm{pk}(L)-k} t^m$$
$$= \frac{1}{2} \left(\frac{1+t}{1-t}\right)^{n+1} \left(\frac{4t}{(1+t)^2}\right)^{1+\mathrm{pk}(L)}$$

which is the generating function of $\theta_{n,L}$ and only depends on n and $\mathrm{pk}\,L$. For a fixed n, these functions are clearly linearly independent for sets L with distinct peak numbers. The result follows immediately from [6, Theorem 4.3]. ☐

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Jinting Liang, Michigan State University, Department of Mathematics, 619 Red Cedar Road, C212 Wells Hall, East Lansing, MI 48824-1027 (USA)
E-mail : liangj26@msu.edu