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Universal characters twisted by roots of unity

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Abstract A classical result of Littlewood gives a factorisation for the Schur function at a set of variables "twisted" by a primitive t-th root of unity, characterised by the core and quotient of the indexing partition. While somewhat neglected, it has proved to be an important tool in the character theory of the symmetric group, the cyclic sieving phenomenon, plethysms of symmetric functions and more. Recently, similar factorisations for the characters of the groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ were obtained by Ayyer and Kumari. We lift these results to the level of universal characters, which has the benefit of making the proofs simpler and the structure of the factorisations more transparent. Our approach also allows for universal character extensions of some factorisations of a different nature originally discovered by Ciucu and Krattenthaler, and generalised by Ayyer and Behrend.

1. Introduction

In his 1940 book *The theory of group characters and matrix representations of groups*, D. E. Littlewood devotes a section to the evaluation of the Schur function s_λ at a set of variables "twisted" (not his term) by a primitive t-th root of unity ζ [27, §7.3]. In modern terminology, Littlewood's theorem asserts that s_λ evaluated at the variables $\zeta^j x_i$ for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant t-1$ is zero unless the t-core of λ is empty. Moreover, when it is nonzero, it factors as a product of Schur functions indexed by the elements of the t-quotient of λ, each with the variables $x_1^t, \dots, x_n^t$.

The Schur functions are characters of the irreducible polynomial representations of the general linear group $\mathrm{GL}(n,\mathbb{C})$. Ayyer and Kumari [5] have recently generalised Littlewood's theorem to the characters of the other classical groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ indexed by partitions. While their factorisations are still indexed by the t-quotient of the corresponding partition, the vanishing is governed by the t-core having a particular form. More precisely, t-core(λ) is of the form $(a \mid a+z)$ in Frobenius notation, where $z=-1,1,0$, for $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ respectively. Note that these are the same partitions occurring in Littlewood's Schur expansion of the Weyl denominators for types B_n, C_n and D_n [27, p. 238] (see also [30, p. 79]).

Littlewood's proof, and the proofs of Ayyer and Kumari, use the Weyl-type expressions for the characters as ratios of alternants. In the Schur case, Chen, Garsia and Remmel [7] and independently Lascoux [23, Theorem 5.8.2] have given an alternate proof based on the Jacobi–Trudi formula (4). This approach was already known to

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Farahat, who used it to extend Littlewood's theorem to skew Schur functions $s_{\lambda/\mu}$ where μ is the t-core of λ [12, Theorem 2]. The full skew Schur case was then given by Macdonald [30, p. 91], again proved using the Jacobi–Trudi formula; see Theorem 3.1 below.

In this article we lift the results of Ayyer and Kumari to the much more general universal characters of the groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ as defined by Koike and Terada [20]. These are symmetric functions indexed by partitions which, under appropriate specialisation of the variables, become actual characters of their respective groups. In fact, these generalise the Jacobi–Trudi-type formulas for the characters of these groups, which were first written down by Weyl [42, Theorems 7.8.E & 7.9.A]. For the universal characters we generalise the notion of "twisting" a set of variables by introducing operators $\varphi_t : \Lambda \longrightarrow \Lambda$ for each integer $t \geqslant 2$ which act on the complete homogeneous symmetric functions as

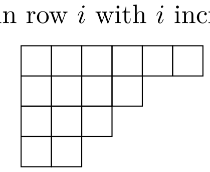
$$\varphi_t h_r = \begin{cases} h_{r/t} & \text{if } t \text{ divides } r, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It is not at all hard to show that the image of φ_t acting on a symmetric function at the variables $x_1^t, \dots, x_n^t$ agrees with the result of twisting the variables $x_1, \dots, x_n$ by ζ. The advantages of this framework for such factorisations are that the proofs are much simpler, and the structure of the factorisations is made transparent. Moreover, we are able to discuss dualities between these objects which are only present at the universal level. A particularly important tool for our purposes is Koike's universal character $\mathrm{rs}_{\lambda,\mu}$ (11) associated with a rational representation of $\mathrm{GL}(n,\mathbb{C})$. This object, which is used later in Subsection 6.3 to prove other character factorisations, appears to be the correct universal character analogue of the Schur function with variables $(x_1, 1/x_1, \dots, x_n, 1/x_n)$.

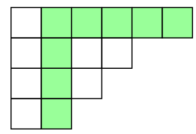
The remainder of the paper reads as follows. In the next section we outline the preliminaries on partitions and symmetric functions needed to state our main results, which follow in Section 3. In the following Section 4 we prepare for the proofs of these results by giving a series of lemmas regarding cores and quotients and their associated signs. The factorisations are then proved in Section 5, including a detailed proof of the Schur case, following Macdonald. The final Section 6 concerns other factorisation results relating to Schur functions and other characters. This includes universal extensions of factorisations very different from those already discussed originally due to Ciucu and Krattenthaler, later generalised by Ayyer and Behrend.

2. Preliminaries

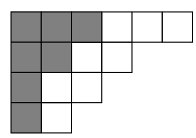
2.1. Partitions. A *partition* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a weakly decreasing sequence of nonnegative integers such that only finitely many of the λ_i are nonzero. The nonzero λ_i are called *parts* and the number of parts the *length*, written $l(\lambda)$. We say λ is *a partition of* n if $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots = n$. Two partitions are regarded as the same if they agree up to trailing zeroes, and the set of all partitions is written $\mathscr{P}$. A partition is identified with its *Young diagram*, which is the left-justified array of squares consisting of λ_i squares in row i with i increasing downward. For example



is the Young diagram of $(6,4,3,2)$. We define the *conjugate* partition λ' by reflecting the diagram of λ in the main diagonal $x = y$, so that the conjugate of $(6,4,3,2)$ above is $(4,4,3,2,1,1)$. If $\lambda=\lambda'$ then λ is called *self-conjugate*. For a square at coordinate (i,j) where $1\leqslant i\leqslant l(\lambda)$ and $1\leqslant j\leqslant \lambda_i$ the *hook length* is $h(i,j)=\lambda_i+\lambda'_j-i-j+1$. For example the square $(1,2)$ in



has hook length 8, with its hook shaded. A partition λ *is a t-core* if it contains no squares of hook length t, the set of which is denoted $\mathscr{C}_t$. For a pair of partitions λ,μ we write $\mu\subseteq\lambda$ if the diagram of μ can be drawn inside the diagram of λ, i.e. if $\mu_i\leqslant\lambda_i$ for all $i\geqslant 1$. In this case we can form the *skew shape* λ/μ by removing the digram of μ from that of λ. For example $(3,2,1,1)\subseteq(6,4,3,2)$ and the diagram of $(6,4,3,2)/(3,2,1,1)$ is given by the non-shaded squares of



A skew shape is called a *ribbon* (or *border strip*, *rim hook*, *skew hook*) if its diagram is connected and contains no 2×2 square. A t-ribbon is a ribbon with t boxes. The *height* of a t-ribbon R, written $\mathrm{ht}(R)$, is one less than the number of rows it occupies. In our example above $R=(6,4,3,2)/(3,2,1,1)$ is an 8-ribbon with height $\mathrm{ht}(R)=3$. We say a skew shape is *tileable by t-ribbons* or *t-tileable* if there exists a sequence of partitions

$$\mu=\nu^{(0)}\subseteq\nu^{(1)}\subseteq\cdots\subseteq\nu^{(k-1)}\subseteq\nu^{(k)}=\lambda \tag{2}$$

such that $\nu^{(i)}/\nu^{(i-1)}$ is a t-ribbon for $1\leqslant i\leqslant k$. A sequence $D=(\nu^{(0)},\dots,\nu^{(k)})$ (not to be confused with the t-quotient of ν below, for which we use the same notation) satisfying (2) is called a *ribbon decomposition* (or *border strip decomposition*) of λ/μ. We define the height of a ribbon decomposition to be the sum of the heights of the individual ribbons: $\mathrm{ht}(D):=\sum_{i=1}^{k}\mathrm{ht}(\nu^{(i)}/\nu^{(i-1)})$. As shown by van Leeuwen [26, Proposition 3.3.1] and Pak [34, Lemma 4.1] (also in [2, §6]), the quantity $(-1)^{\mathrm{ht}(D)}$ is the same for every ribbon decomposition of λ/μ. We therefore define the *sign* of a t-tileable skew shape λ/μ as

$$\mathrm{sgn}_t(\lambda/\mu):=(-1)^{\mathrm{ht}(D)}. \tag{3}$$

Let $\mathrm{rk}(\lambda)$ be the greatest integer such that $\mathrm{rk}(\lambda)\geqslant\lambda_{\mathrm{rk}(\lambda)}$, usually called the *Frobenius rank* of λ. Equivalently, $\mathrm{rk}(\lambda)$ is the side length of the largest square which fits inside the diagram of λ (the *Durfee square*). A partition can alternatively be written in *Frobenius notation* as

$$\lambda=\big(\lambda_1-1,\dots,\lambda_{\mathrm{rk}(\lambda)}-\mathrm{rk}(\lambda)\mid\lambda'_1-1,\dots,\lambda'_{\mathrm{rk}(\lambda)}-\mathrm{rk}(\lambda)\big).$$

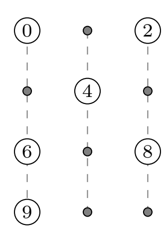
Any pair of integer sequences $a_1>\cdots>a_k\geqslant 0$ and $b_1>\cdots>b_k\geqslant 0$ thus determines a partition $\lambda=(a\mid b)$ with $\mathrm{rk}(\lambda)=k$. For $z\in\mathbb{Z}$ and an integer sequence of predetermined length $a=(a_1,\dots,a_k)$ we write $a+z:=(a_1+z,\dots,a_k+z)$. Following Ayyer and Kumari [5, Definition 2.9], λ is called *z-asymmetric* if it is of the form $\lambda=(a\mid a+z)$ for some integer sequence a and integer z. Clearly a 0-asymmetric partition is self-conjugate. Partitions which are -1- and 1-asymmetric are called *orthogonal* and *symplectic* respectively.

2.2. Cores and quotients. We now describe the t-core and t-quotient of λ arithmetically following [30, p. 12]. There are many equivalent descriptions, see for instance [13, 15, 16, 41]. We begin with the *beta set* of a partition, which is simply the set of n integers

$$\beta(\lambda;n) := \{\lambda_1+n-1, \lambda_2+n-2, \ldots, \lambda_{n-1}+1, \lambda_n\},$$

where $n \geqslant l(\lambda)$ is fixed. The number of elements in this set congruent to r modulo t is denoted by $m_r(\lambda;n) = m_r$. An element which falls into residue class r for $0 \leqslant r \leqslant t-1$ can be written as $\xi_k^{(r)}t+r$ for some integers $\xi_1^{(r)} > \cdots > \xi_{m_r}^{(r)} \geqslant 0$. These integers are used to define a partition with parts $\lambda_k^{(r)} = \xi_k^{(r)} - m_r(\lambda;n)+k$ where $1 \leqslant k \leqslant m_r(\lambda;n)$, and the ordered sequence $(\lambda^{(0)},\ldots,\lambda^{(t-1)})$ of these partitions is called the *t-quotient*. The precise order of the constituents of the t-quotient depends on the residue class of n modulo t. However, the orders only differ by cyclic permutations, and Macdonald comments that it is best to think of the quotient as a sort-of "necklace" of partitions. To simplify things somewhat, we adopt the convention that the t-quotient is always computed with n a multiple of t, so that the order of its constituents is fixed. To define the t-core, one writes down the n distinct integers $kt+r$ where $0 \leqslant k \leqslant m_r(\lambda;n)-1$ and $0 \leqslant r \leqslant t-1$ in descending order, say as $\tilde{\xi}_1 > \cdots > \tilde{\xi}_n$. Then $t\text{-core}(\lambda)_i := \tilde{\xi}_i - n + i$. If t-core(λ) is empty then we say λ *has empty t-core*.

It will prove useful later on to work with the *bead configurations* (or *bead diagrams*, *abacus model*) of James and Kerber [16, §2.7], which give a different model for t-cores and t-quotients. The "board" for a bead configuration is the set of nonnegative integers arranged in t downward-increasing columns, called *runners*, according to their residues modulo t. A bead is then placed at the space corresponding to each element of $\beta(\lambda;n)$. For an example, let $\lambda = (4,4,3,2,1)$ so that $\beta(\lambda;6) = \{9,8,6,4,2,0\}$. Then the bead configuration for λ with $t=3$ and $n=6$ and the beads labelled by their position is



Moving a bead up one space is equivalent to reducing one of the elements of $\beta(\lambda;n)$ by t. This is, in turn, equivalent to removing a t-ribbon from λ such that what remains is still a Young diagram (see for instance [30, p. 12]). Pushing all beads to the top will give the bead configuration of t-core(λ), and this is clearly independent of the order in which the beads are pushed. It follows that t-core(λ) is the unique partition obtained by removing t-ribbons (in a valid way) from the diagram of λ until it is no longer possible to do so. We note that if removing a ribbon R corresponds to moving a bead from position b to $b-t$, then $\mathrm{ht}(R)$ is equal to the number of beads lying at the positions strictly between $b-t$ and b. The t-quotient can be obtained from the bead configuration by reading the r-th runner, bottom-to-top, as a bead configuration with $m_r(\lambda;n)$ beads. For our example, this means that $\beta(t\text{-core}(\lambda);6) = \{6,5,3,2,1,0\}$, so that t-core$(\lambda) = (1,1)$, with the quotient $((1,1),(1),(1))$ computed similarly.

The above procedure of computing the t-core and t-quotient actually encodes a bijection

$$\begin{aligned} \phi_t : \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)},\ldots,\lambda^{(t-1)})\big), \end{aligned}$$

such that $|\lambda| = |t\text{-core}(\lambda)| + t(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}|)$. The arithmetic description of this correspondence was first written down by Littlewood [29]. The idea of removing ribbons from a partition until a unique core is obtained goes back to Nakayama [32]. The t-quotient of a partition has its origin in the *star diagrams* of Nakayama, Osima, Robinson and Staal [33, 38, 39], which were shown to be equivalent to Littlewood's t-quotient by Farahat [10].

Let $w_t(\lambda; n)$ be the permutation of $\beta(\lambda; n)$ which sorts the elements of the beta set so that their residues modulo t are increasing, and the elements within each residue class decrease. The sign of $w_t(\lambda; n)$ will be denoted $\mathrm{sgn}(w_t(\lambda; n))$. The permutation $w_t(\lambda; n)$ can also be read off the bead configuration by first labelling the beads "backwards": label the bead with largest place 1, second-largest 2, and so on. Reading the labels column-wise from bottom-to-top gives $w_t(\lambda; n)$ in one-line notation. An inversion in this permutation corresponds to a pair of beads b_1, b_2 such that b_2 lies weakly below and strictly to the right of b_1. With the same example as before $w_3((4,4,3,2,1);6) = 136425$ and the bead at position 0 generates three inversions, as it "sees" the beads $2, 4$ and 8.

As follows from the above, a partition λ has empty t-core if and only if it is t-tileable. In our results below we will need the following characterisation of when a skew shape is t-tileable, generalising the notion of "empty t-core" to this case. We briefly recall our convention that t-quotients are always computed with the number of beads in the bead configuration a multiple of t.

Lemma 2.1. *A skew shape λ/μ is tileable by t-ribbons if and only if t-core$(\lambda) = t$-core(μ) and $\mu^{(r)} \subseteq \lambda^{(r)}$ for each $0 \leqslant r \leqslant t-1$.*

Proof. The skew shape being t-tileable is equivalent to the diagram of μ being obtainable from the diagram of λ by removing t-ribbons. In other words, we can obtain the bead configuration of μ from that of λ, where both have nt beads, by moving beads upwards. Assume that this is the case. Then $m_r(\lambda; nt) = m_r(\mu; nt)$ for each $0 \leqslant r \leqslant t-1$, so that the r-th runner has the same number of beads in each diagram. This implies that t-core$(\lambda) = t$-core(μ). It also follows that the i-th bead in each runner of λ's bead configuration must lie weakly below the i-th bead in the same runner of μ's bead configuration. Equivalently, $\mu_i^{(r)} \leqslant \lambda_i^{(r)}$ for all $0 \leqslant r \leqslant t-1$ and $1 \leqslant i \leqslant m_r(\lambda; nt)$, which in turn is equivalent to $\mu^{(r)} \subseteq \lambda^{(r)}$. The reverse direction is now clear. ☐

Note that the lemma is also true when the t-quotients of λ and μ are computed using the same integer n of any residue class modulo t. If λ/μ is t-tileable, then we think of $\lambda^{(0)}/\mu^{(0)}, \dots, \lambda^{(t-1)}/\mu^{(t-1)}$ as its t-quotient. When λ/μ is not t-tileable, it is not so clear how to define the t-quotient.

2.3. Symmetric functions and universal characters. Here we discuss some basics of the theory of symmetric functions, following [30]. Let Λ denote the *ring of symmetric functions* in an arbitrary countable set of variables $X = (x_1, x_2, x_3, \dots)$, called an *alphabet*. Where possible, we write elements of Λ without reference to an alphabet if the expression is independent of the chosen alphabet. If for a positive integer n one sets $x_i = 0$ for all $i > n$ then the elements of Λ reduce to symmetric polynomials in the variables $(x_1, \dots, x_n)$. Another common specialisation sets $x_{n+i} = x_i^{-1}$ for $1 \leqslant i \leqslant n$ and $x_i = 0$ for $i > 2n$. This gives Laurent polynomials in the x_i invariant under permutation and inversion of the variables (i.e., BC_n-symmetric functions). We will later write $(x_1^{\pm}, \dots, x_n^{\pm})$ for this alphabet.

Two fundamental algebraic bases for Λ are the *complete homogeneous symmetric functions* and the *elementary symmetric functions*, defined for any positive integer r

by

$$h_r(X) := \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_r} x_{i_1}\cdots x_{i_r} \quad \text{and} \quad e_r(X) := \sum_{1\leqslant i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r},$$

respectively. We further set $h_0 = e_0 := 1$ and $h_{-r} = e_{-r} = 0$ for positive r. These admit the generating functions

$$\begin{aligned} H_z(X) &:= \sum_{r\geqslant 0} z^r h_r(X) = \prod_{i\geqslant 1}\frac{1}{1-zx_i} \\ E_z(X) &:= \sum_{r\geqslant 0} z^r e_r(X) = \prod_{i\geqslant 1}(1+zx_i). \end{aligned}$$

The h_r and e_r for $r \geqslant 1$ are algebraically independent over $\mathbb{Z}$ and generate Λ. In view of this, we can define a homomorphism $\omega : \Lambda \longrightarrow \Lambda$ by $\omega h_r = e_r$. It then follows from the relation $H_z(X)E_{-z}(X) = 1$ that $\omega e_r = h_r$, so that ω is an involution. We also define the power sums by

$$p_r(X) := \sum_{i\geqslant 1} x_i^r,$$

for $r \geqslant 1$ and $p_0 := 1$. These satisfy $\omega p_r = (-1)^{r-1}p_r$.

The most important family of symmetric functions are the *Schur functions*. These have several definitions, but for our purposes it is best to define them, already for skew shapes, by the *Jacobi–Trudi formula*. If λ/μ is a skew shape and n an integer such that $n \geqslant l(\lambda)$ we define

$$s_{\lambda/\mu} := \det_{1\leqslant i,j\leqslant n}(h_{\lambda_i-\mu_j-i+j}). \tag{4}$$

This is independent of n as long as $n \geqslant l(\lambda)$. If $\mu \not\subseteq \lambda$ then we set $s_{\lambda/\mu} := 0$. There is also an equivalent formula in terms of the e_r, called the *dual Jacobi–Trudi formula* (rarely also the *Nägelsbach–Kostka identity*)

$$s_{\lambda/\mu} = \det_{1\leqslant i,j\leqslant m}(e_{\lambda'_i-\mu'_j-i+j}).$$

Restricting to the μ empty case, we have $s_{(r)} = h_r$ and $s_{(1^r)} = e_r$. Moreover, it is clear that $\omega s_{\lambda/\mu} = s_{\lambda'/\mu'}$.

If the set of variables $(x_1,\dots,x_n)$ is finite then the Schur function for $\mu = 0$ admits another definition as a ratio of alternants

$$s_\lambda(x_1,\dots,x_n) = \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})}. \tag{5}$$

The denominator is the Vandermonde determinant and has the product representation $\det_{1\leqslant i,j\leqslant n}(x_i^{n-j}) = \prod_{1\leqslant i<j\leqslant n}(x_i-x_j)$. In this case we also define $s_\lambda(x_1,\dots,x_n) = 0$ if $l(\lambda) > n$. If λ is a partition of length at most n, then

$$s_{(\lambda_1+1,\dots,\lambda_n+1)}(x_1,\dots,x_n) = (x_1\cdots x_n)s_{(\lambda_1,\dots,\lambda_n)}(x_1,\dots,x_n). \tag{6}$$

This allows for Schur functions with a finite set of n variables to be extended to weakly decreasing sequences of integers of length exactly n.

Following Koike and Terada we define the *universal characters* for $\mathrm{O}(2n,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ as the symmetric functions [20, Definition 2.1.1]

$$\mathrm{o}_\lambda := \det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-i+j} - h_{\lambda_i-i-j}\big) \tag{7}$$

$$\mathrm{sp}_\lambda := \frac{1}{2}\det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2}\big), \tag{8}$$

where $n \geqslant l(\lambda)$. Like the Schur functions, these determinants also have dual versions

$$
\begin{aligned}
\mathrm{o}_\lambda &= \frac{1}{2} \det_{1 \leqslant i,j \leqslant m} \left(e_{\lambda'_i - i + j} + e_{\lambda'_i - i - j + 2} \right) \\
\mathrm{sp}_\lambda &= \det_{1 \leqslant i,j \leqslant m} \left(e_{\lambda'_i - i + j} - e_{\lambda'_i - i - j} \right),
\end{aligned} \tag{9}
$$

where $m \geqslant \lambda_1$. From this it is clear that $\omega \mathrm{o}_\lambda = \mathrm{sp}_{\lambda'}$. Koike alone added a third universal character for the group $\mathrm{SO}(2n+1, \mathbb{C})$ [19, Definition 6.4] (see also [25, Equation (3.8)])

$$
\mathrm{so}_\lambda := \det_{1 \leqslant i,j \leqslant n} \left(h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 1} \right) = \det_{1 \leqslant i,j \leqslant m} \left(e_{\lambda'_i - i + j} + e_{\lambda'_i - i - j + 1} \right).
$$

This universal character is self-dual under ω, so $\omega \mathrm{so}_\lambda = \mathrm{so}_{\lambda'}$. For later convenience we also define a variant of the above as

$$
\mathrm{so}^-_\lambda := \det_{1 \leqslant i,j \leqslant n} \left(h_{\lambda_i - i + j} - h_{\lambda_i - i - j + 1} \right) = \det_{1 \leqslant i,j \leqslant m} \left(e_{\lambda'_i - i + j} - e_{\lambda'_i - i - j + 1} \right).
$$

If $X = (x_1, x_2, x_3, \dots)$ is a set of variables (which may be finite or countable) and $-X := (-x_1, -x_2, -x_3, \dots)$, then

$$
\mathrm{so}^-_\lambda(X) = (-1)^{|\lambda|} \mathrm{so}_\lambda(-X),
$$

since the h_r and e_r are homogeneous of degree r.

For $l(\lambda) \leqslant n$, each of the three above universal characters become actual characters of irreducible representations of their associated groups when specialised to $(x_1^{\pm}, \dots, x_n^{\pm})$ (hence the name universal characters).

The irreducible polynomial representations of $\mathrm{GL}(n, \mathbb{C})$ are indexed by partitions of length at most n. On the other hand, the irreducible rational representations are indexed by weakly decreasing sequences of integers of length n, which are called *staircases* by Stembridge [40]. Such sequences are equivalent to pairs of partitions λ, μ such that $l(\lambda) + l(\mu) \leqslant n$. Given such a pair, one defines the associated staircase $[\lambda, \mu]$ by $[\lambda, \mu]_i := \lambda_i - \mu_{n-i+1}$ for $1 \leqslant i \leqslant n$. The characters of the rational representations of $\mathrm{GL}(n, \mathbb{C})$ are then given by $s_{[\lambda,\mu]}(x_1, \dots, x_n)$ for all staircases with n entries. Note that (6) implies that this object is just a Schur function up to a power of $x_1 \cdots x_n$. In [28], Littlewood gave the expansion

$$
s_{[\lambda,\mu]}(x_1, \dots, x_n) = \sum_\nu (-1)^{|\nu|} s_{\lambda/\nu}(x_1, \dots, x_n) s_{\mu/\nu'}(1/x_1, \dots, 1/x_n). \tag{10}
$$

For a pair of partitions λ, μ and sets of indeterminates X, Y, this may be used to define the universal character associated to a rational representation of $\mathrm{GL}(n, \mathbb{C})$ as

$$
\mathrm{rs}_{\lambda,\mu}(X; Y) := \sum_\nu (-1)^{|\nu|} s_{\lambda/\nu}(X) s_{\mu/\nu'}(Y). \tag{11}
$$

Note that the only terms which contribute are those with $\nu \subseteq \lambda$ and $\nu' \subseteq \mu$. If we let ω_X and ω_Y denote the involution ω acting on the set of variables in its subscript, then

$$
\begin{aligned}
\omega_X \omega_Y \mathrm{rs}_{\lambda,\mu}(X; Y) &= \sum_\nu (-1)^{|\nu|} s_{\lambda'/\nu'}(X) s_{\mu'/\nu}(Y) \\
&= \sum_{\nu'} (-1)^{|\nu|} s_{\lambda'/\nu}(X) s_{\mu'/\nu'}(Y) \\
&= \mathrm{rs}_{\lambda',\mu'}(X; Y).
\end{aligned}
$$

As shown by Koike [18], this object has a Jacobi–Trudi-type expression as a block matrix

$$\mathrm{rs}_{\lambda,\mu}(X;Y) = \det_{1\leqslant i,j\leqslant n+m} \begin{pmatrix} \big(h_{\lambda_i-i+j}(X)\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}(X)\big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} \\ \big(h_{\mu_i-i-j+1}(Y)\big)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} & \big(h_{\mu_i-i+j}(Y)\big)_{1\leqslant i,j\leqslant m} \end{pmatrix}, \tag{12}$$

where $n \geqslant l(\lambda)$ and $m \geqslant l(\mu)$. As for the other determinants, this is independent of n and m as long as $n \geqslant l(\lambda)$ and $m \geqslant l(\mu)$. The relation under $\omega_X\omega_Y$ implies we have the dual form [18, Definition 2.1]

$$\mathrm{rs}_{\lambda,\mu}(X;Y) = \det_{1\leqslant i,j\leqslant n+m} \begin{pmatrix} \big(e_{\lambda'_i-i+j}(X)\big)_{1\leqslant i,j\leqslant n} & \big(e_{\lambda'_i-i-j+1}(X)\big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} \\ \big(e_{\mu'_i-i-j+1}(Y)\big)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} & \big(e_{\mu'_i-i+j}(Y)\big)_{1\leqslant i,j\leqslant m} \end{pmatrix}, \tag{13}$$

where $n \geqslant \lambda_1$ and $m \geqslant \mu_1$. The definition (11) and the determinants (12) and (13) are related by taking the Laplace expansion of each determinant according to its presented block structure; see [18, Equation (2.1)]. Also, by the definition (11) and (10) it is immediate that, for $l(\lambda) + l(\mu) \leqslant n$,

$$\mathrm{rs}_{\lambda,\mu}(x_1,\dots,x_n;1/x_1,\dots,1/x_n) = s_{[\lambda,\mu]}(x_1,\dots,x_n).$$

We will always take $X = Y$ in $\mathrm{rs}_{\lambda,\mu}(X;Y)$, which we write as $\mathrm{rs}_{\lambda,\mu}$ in the rest of the paper. In particular we note that $\mathrm{rs}_{\lambda,\mu} = \mathrm{rs}_{\mu,\lambda}$.

As mentioned in the introduction, the notion of twisting the set of variables $x_1,\dots,x_n$ by a primitive t-th root of unity ζ is replaced by the operator φ_t (1), which has been considered by Macdonald [30, p. 91] and, for $t = 2$, by Baik and Rains [6, p. 25]. Let $X^t := (x_1^t, x_2^t, x_3^t \dots)$ and denote by ψ_t the homomorphism

$$\begin{aligned} \psi_t : \Lambda &\longrightarrow \Lambda_{X^t} \\ f &\longmapsto f(X, \zeta X, \dots, \zeta^{t-1}X). \end{aligned}$$

Since $\psi_t H_z(X) = H_{z^t}(X^t)$, both φ_t and ψ_t act on the h_r in the same way, i.e. the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\varphi_t"] \arrow[dr, "\psi_t"'] & \Lambda \arrow[d, "X^t"] \\ & \Lambda_{X^t} \end{tikzcd} \tag{14}$$

commutes, where the arrow labelled X^t is the substitution map. This implies the claim of the introduction that the action of φ_t is equivalent to twisting the alphabet X by a primitive t-th root of unity ζ. If one wishes to think about this as a map $\Lambda_X \longrightarrow \Lambda_X$ where X is some concrete alphabet, then substitute each $x \in X$ by its set of t-th roots $x^{1/t}, \zeta x^{1/t}, \dots, \zeta^{t-1}x^{1/t}$ and evaluate this expression. By the action of this map on the h_r, such a map gives a symmetric function again in the variables X. Using the generating function $E_z(X)$ one may also show that [23, §5.8]

$$\varphi_t e_r = \begin{cases} (-1)^{r(t-1)/t} e_{r/t} & \text{if } t \text{ divides } r, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore ω and φ_t commute if t is odd, but not in general. Proposition 3.5 shows that, in some cases, the maps commute up to a computable sign. Different, but closely related operators are discussed at the end of this paper.

3. Summary of results

With the preliminary material of the previous section under our belts, we are now ready to state our main results regarding factorisations of universal characters under φ_t. The first of these is the action of the map on the skew Schur functions.

Theorem 3.1. *We have that $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is tileable by t-ribbons, in which case*

$$\varphi_t s_{\lambda/\mu} = \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}.$$

For μ empty, this result is due to Littlewood [27, p. 131], who proves it by direct manipulation of the ratio of alternants (5). By (14) with $X = (x_1, \dots, x_n)$, one can recover Littlewood's result by simply evaluating the right-hand side of the equation at $(x_1^t, \dots, x_n^t)$. He also states the μ empty case of the theorem in the language of symmetric group characters: see both [27, p. 144] and [29, p. 340]. The generalisation to skew characters was discovered by Farahat [11] (see also [9, Theorem 3.3]). The form we state here is precisely that of Macdonald [30, p. 91]. Curiously, Prasad recently rediscovered the μ empty case independently, with a proof identical to Littlewood's, but in a more representation-theoretic context [36]. A version of the result for Schur's P-and Q-functions has been given by Mizukawa [31, Theorem 5.1].

Theorem 3.1 has been rediscovered many times for both skew and straight shapes, and often only in special cases. We make no attempt to give a complete history, but it appears to us that the theorem deserves to be better known. The interested reader can consult [43] for some exposition on the character theory side of this story. On the symmetric functions side, such an exposition is lacking in the literature.

We now state, in sequence, the three factorisations lifting [5, Theorems 2.11, 2.15 & 2.17] to the level of universal characters, beginning with the universal orthogonal character.

Theorem 3.2. *Let λ be a partition of length at most nt. Then $\varphi_t \mathrm{o}_\lambda = 0$ unless t-core(λ) is orthogonal, in which case*

$$\varphi_t \mathrm{o}_\lambda = (-1)^{\varepsilon^{\mathrm{o}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda; nt)) \mathrm{o}_{\lambda^{(0)}} \prod_{r=1}^{\lfloor (t-1)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(t/2)}} & t \text{ even}, \\ 1 & t \text{ odd}, \end{cases}$$

where

$$\varepsilon^{\mathrm{o}}_{\lambda;nt} = \sum_{r=\lfloor (t+2)/2 \rfloor}^{t-1} \binom{m_r(\lambda; nt)+1}{2} + \mathrm{rk}(t\text{-core}(\lambda)) + \begin{cases} \binom{n+1}{2} + n\,\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ even}, \\ 0 & t \text{ odd}. \end{cases}$$

Our next result is the same factorisation for the symplectic character.

Theorem 3.3. *Let λ be a partition of length at most nt. Then $\varphi_t \mathrm{sp}_\lambda = 0$ unless t-core(λ) is symplectic, in which case*

$$\varphi_t \mathrm{sp}_\lambda = (-1)^{\varepsilon^{\mathrm{sp}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda; nt)) \mathrm{sp}_{\lambda^{(t-1)}} \prod_{r=0}^{\lfloor (t-3)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-2)}} \times \begin{cases} \mathrm{so}_{\lambda^{((t-2)/2)}} & t \text{ even}, \\ 1 & t \text{ odd}, \end{cases}$$

where

$$\varepsilon^{\mathrm{sp}}_{\lambda;nt} = \sum_{r=\lfloor t/2 \rfloor}^{t-2} \binom{m_r(\lambda; nt)+1}{2} + \begin{cases} \binom{n+1}{2} + n\,\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ even}, \\ 0 & t \text{ odd}. \end{cases}$$

Finally, we can claim a similar factorisation for so_λ.

Theorem 3.4. *Let λ be a partition of length at most nt. Then $\varphi_t \mathrm{so}_\lambda = 0$ unless t-core(λ) is self-conjugate, in which case*

$$\varphi_t \mathrm{so}_\lambda = (-1)^{\varepsilon^{\mathrm{so}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda;nt)) \prod_{r=0}^{\lfloor (t-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-1)}} \times \begin{cases} 1 & t \text{ even}, \\ \mathrm{so}_{\lambda^{((t-1)/2)}} & t \text{ odd}, \end{cases}$$

where(1)

$$\varepsilon^{\mathrm{so}}_{\lambda;nt} = \sum_{r=\lfloor (t+1)/2 \rfloor}^{t-1} \binom{m_r(\lambda;nt)+1}{2} + \begin{cases} 0 & t \text{ even}, \\ n \mathrm{rk}(t\text{-core}(\lambda)) & t \text{ odd}. \end{cases}$$

Some remarks are in order. Firstly, the three signs $\operatorname{sgn}(w_t(\lambda;nt))(-1)^{\varepsilon^{\bullet}_{\lambda;nt}}$ are actually independent of n as long as $nt \geqslant l(\lambda)$, a fact which we prove in Lemma 4.8 below. As remarked by Ayyer and Kumari [5, Remark 2.19], the order of the quotient is unchanged upon replacing $n \mapsto n+1$, so the product in the evaluation is independent of n. It is in principle possible to carry out our proof technique below under the assumption that $l(\lambda)$ is bounded by an arbitrary integer, say k, where k is not necessarily a multiple of t. In this case the evaluation is of course the same, however the sign will be expressed differently and the t-quotients in the evaluations will be a cyclic permutation of the ones presented. Since the proof is simplest when this k is a multiple of t, we stick to this case.

To obtain the theorems of Ayyer and Kumari one evaluates the right-hand side of each identity at the set of variables $(x_1^{\pm t}, \dots, x_n^{\pm t})$. Using (6) and the definition of $\mathrm{rs}_{\lambda,\mu}$ it follows that in this case the rational universal characters occurring in each evaluation agree with the Schur functions $s_{\mu_i^{(k)}}(x_1^{\pm t}, \dots, x_n^{\pm t})$ in the notation of [5].

As we have already seen the maps ω and φ_t do not commute in general. However, when acting on $s_{\lambda/\mu}$ and so_λ, they commute up to an explicitly computable sign.

Proposition 3.5. *We have the relations*

$$\omega \varphi_t s_{\lambda/\mu} = (-1)^{(t-1)(|\lambda^{(0)}/\mu^{(0)}| + \cdots + |\lambda^{(t-1)}/\mu^{(t-1)}|)} \varphi_t \omega s_{\lambda/\mu},$$

and

$$\omega \varphi_t \mathrm{so}_\lambda = (-1)^{(t-1)(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}|)} \varphi_t \omega \mathrm{so}_\lambda.$$

We remark that the second relation does not hold with so_λ replaced by sp_λ or o_λ as written above since $\omega \mathrm{so}^-_\lambda = \mathrm{so}^-_{\lambda'}$.

4. Auxiliary results

The purpose of this section is to collect all the small facts about beta sets and the signs (3) which we need to prove our main results. To begin, we relate the bead configurations of a partition and its conjugate.

Lemma 4.1. *Let λ be a partition of length at most nt such that $\lambda_1 \leqslant mt$. Then the bead configuration for $\beta(\lambda';mt)$ can be obtained from the bead configuration for $\beta(\lambda;nt)$ with $n+m$ rows by rotating the picture by 180° and then interchanging beads and spaces.*

Proof. This is a consequence of the fact [30, p. 3] that for $l(\lambda) \leqslant n$ and $\lambda_1 \leqslant m$,

$$\{0,1,\dots,m+n-1\} = \{\lambda_i + n - i : 1 \leqslant i \leqslant n\} \sqcup \{m+n-1-(\lambda'_j + m - j) : 1 \leqslant j \leqslant m\},$$

where $\sqcup$ denotes a disjoint union. $\square$

(1)We have corrected the lower bound in the sum defining $\varepsilon^{\mathrm{so}}_{\lambda;nt}$ from $\lfloor t/2 \rfloor$ in [5, Theorem 2.17] (there denoted ϵ) to $\lfloor (t+1)/2 \rfloor$.

This lemma immediately implies the following relationship between the t-core and t-quotient of λ and λ'.

Corollary 4.2. *For a partition λ we have $t\text{-core}(\lambda') = t\text{-core}(\lambda)'$ and the t-quotient of λ' is $\big((\lambda^{(t-1)})', \dots, (\lambda^{(0)})'\big)$.*

The next pair of lemmas are due to Ayyer and Kumari, the first of which characterises partitions with z-asymmetric t-cores in terms of their beta sets [5, Lemma 3.6].

Lemma 4.3. *For a partition λ of length at most nt, $t\text{-core}(\lambda)$ is of the form $(a \mid a+z)$ for some integer $-1 \leqslant z \leqslant t-1$ if and only if*

$$m_r(\lambda, nt) + m_{t-r-z-1}(\lambda, nt) = 2n \quad \text{for } 0 \leqslant r \leqslant t-z-1, \tag{15a}$$

$$m_r(\lambda, nt) = n \quad \text{for } t-z \leqslant r \leqslant t-1, \tag{15b}$$

where the indices of the m_r are taken modulo t.

The second lemma of Ayyer and Kumari we need is [5, Lemma 3.13], which is used later on to simplify signs.

Lemma 4.4. *Let λ be a partition of length at most nt. If $t\text{-core}(\lambda)$ is orthogonal, then*

$$\operatorname{rk}(t\text{-core}(\lambda)) = \sum_{r=1}^{\lfloor (t-1)/2 \rfloor} |m_r(\lambda; nt) - n| = \sum_{r=\lfloor (t+2)/2 \rfloor}^{t-1} |m_r(\lambda; nt) - n| \tag{16}$$

If $t\text{-core}(\lambda)$ is symplectic, then

$$\operatorname{rk}(t\text{-core}(\lambda)) = \sum_{r=0}^{\lfloor (t-3)/2 \rfloor} |m_r(\lambda; nt) - n| = \sum_{r=\lfloor t/2 \rfloor}^{t-2} |m_r(\lambda; nt) - n| \tag{17}$$

If $t\text{-core}(\lambda)$ is self-conjugate, then

$$\operatorname{rk}(t\text{-core}(\lambda)) = \sum_{r=0}^{\lfloor (t-2)/2 \rfloor} |m_r(\lambda; nt) - n| = \sum_{r=\lfloor (t+1)/2 \rfloor}^{t-1} |m_r(\lambda; nt) - n| \tag{18}$$

Next, we show that the sign of a tileable skew shape can be expressed in terms of the signs of the permutations $w_t(\lambda; n)$.

Lemma 4.5. *For λ/μ t-tileable and any integer n such that $n \geqslant l(\lambda)$,*

$$\operatorname{sgn}_t(\lambda/\mu) = \operatorname{sgn}(w_t(\lambda; n)) \operatorname{sgn}(w_t(\mu; n)).$$

Proof. Since λ/μ is t-tileable, it has a ribbon decomposition $D = (\nu^{(0)}, \dots, \nu^{(k)})$ where $\nu^{(0)} = \mu$ and $\nu^{(k)} = \lambda$. Also, $\nu^{(k-1)}$ can be obtained from λ by moving one bead at some position upward one space. By our characterisation of the inversions in the permutation $w_t(\lambda; n)$, we see that moving a bead at position ℓ up one space changes the sign by $(-1)^{b_k}$ where b_k is the number of beads at positions between $\ell - t$ and ℓ. In other words, $\operatorname{sgn}(w_t(\lambda; n)) = (-1)^{b_k} \operatorname{sgn}(w_t(\nu^{(k-1)}; n))$. Moreover, $b_k = \operatorname{ht}(\nu^{(k)}/\nu^{(k-1)})$, so that

$$\operatorname{sgn}(w_t(\lambda; n)) \operatorname{sgn}(w_t(\mu; n)) = (-1)^{\sum_{i=1}^k b_i} = (-1)^{\operatorname{ht}(D)} = \operatorname{sgn}_t(\lambda/\mu). \qquad \square$$

We also have the following useful relationship between the sign of λ/μ and λ'/μ'.

Lemma 4.6. *For λ/μ t-tileable,*

$$\operatorname{sgn}_t(\lambda/\mu) \operatorname{sgn}_t(\lambda'/\mu') = (-1)^{(t-1)(|\lambda^{(0)}| + \dots + |\lambda^{(t-1)}| - |\mu^{(0)}| - \dots - |\mu^{(t-1)}|)}.$$

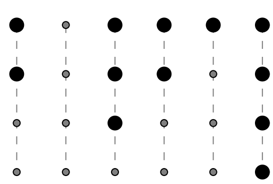
Proof. To prove the claim of the lemma we will proceed by induction on $|\lambda/\mu|$. If $|\lambda/\mu| = 0$ then $\lambda = \mu$ and the equation is trivial. Now fix μ and assume the result holds for λ/μ being t-tileable. Adding a t-ribbon to λ/μ moves one of the beads, say at position b, in the bead configuration for λ down a single space. The change in the number of inversions in $w_t(\lambda; nt)$ is the number of beads b' such that $b < b' < b+1$. A consequence of Lemma 4.1 is that $w_t(\lambda'; mt)$ will change by the number of empty spaces between b and $b+1$. There are $t-1$ spaces and beads between b and $b+1$, so the left-hand side changes by $(-1)^{t-1}$ when adding a t-ribbon. But adding a t-ribbon to λ/μ changes some element of the t-quotient of λ by a single box, also corresponding to a change in sign of $(-1)^{t-1}$. ☐

There is another sign relation between orthogonal and symplectic t-cores, but this time using the permutations w_t.

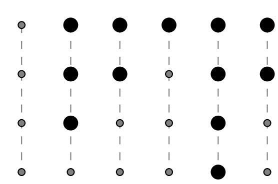
Lemma 4.7. *Let λ be an orthogonal or symplectic t-core whose diagram is contained in an $nt \times nt$ square. Then*

$$\operatorname{sgn}(w_t(\lambda; nt)) \operatorname{sgn}(w_t(\lambda'; nt)) = (-1)^{\operatorname{rk}(\lambda)}.$$

Proof. Assume that λ is a non-empty, orthogonal t-core (if λ is empty the result is trivial) and fix n so that the condition of the theorem holds. The key observation is that for an orthogonal t-core, the bead configuration of λ' with nt beads can be obtained from the bead configuration of λ with nt beads by reducing the labels by 1 modulo t. For example if $\lambda = (12, 7, 5, 3, 2, 2, 1, 1, 1, 1, 1)$ then $\lambda' = (11, 6, 4, 3, 3, 2, 2, 1, 1, 1, 1, 1)$ and their bead configurations for $t = 6$ and $n = 2$ are



and



respectively, where we have suppressed the labels. This is a consequence of Lemma 4.3 with $z = \pm 1$ and Lemma 4.1. When passing from λ to λ', the inversions contributed by the beads in the first runner are removed and replaced by additional inversions associated to the remaining beads in the first n rows. Modulo two, this is equivalent to each bead in the zeroth runner now seeing all of the beads in the same row twice, plus all other beads in the other runners once. Let b be the number of beads in the first n rows of the runners from 1 to $t-1$ in the bead configuration of λ. Then the sign change is

$$\operatorname{sgn}(w_t(\lambda; nt)) = \operatorname{sgn}(w_t(\lambda'; nt))(-1)^{n^2(t-1)+b}.$$

Since λ is an orthogonal t-core, $n^2(t-1) + b \equiv \operatorname{rk}(\lambda) \pmod 2$ by (16) and Lemma 4.3 with $z = -1$. ☐

The next lemma proves the claim made after Theorems 3.2–3.4 that the signs occurring in those factorisations are independent of n.

Lemma 4.8. *The signs $(-1)^{\varepsilon^{\bullet}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda; nt))$ for $\bullet \in \{\mathrm{o}, \mathrm{sp}, \mathrm{so}\}$ are independent of n as long as $nt \geqslant l(\lambda)$.*

Proof. Assume that $nt \geqslant l(\lambda)$. Incrementing n by one adds a row of beads to the top of the bead configuration of λ, and so $m_r(\lambda; (n+1)t) = m_r(\lambda; nt) + 1$. In the inversion count, the rth bead in the new first row sees

$$\sum_{k=r+1}^{t-1} (m_k(\lambda; nt) + 1)$$

other beads. Summing over $k = 0, \ldots, t-1$ we see that

$$\operatorname{sgn}(w_t(\lambda;(n+1)t)) = \operatorname{sgn}(w_t(\lambda;nt))(-1)^{\sum_{r=1}^{\lfloor t/2\rfloor}(m_{2r-1}(\lambda;nt)+1)}.$$

Now assume that λ has an orthogonal t-core. Then by Lemma 4.3 with $z=-1$ the above has the same parity as

$$\sum_{r=1}^{\lfloor t/2\rfloor}(m_{2r-1}(\lambda;nt)+1) \equiv \begin{cases} \frac{(n+1)t}{2} & t \text{ even},\\ \frac{t-1}{2} + \sum_{r=1}^{(t-1)/2} m_r(\lambda;nt) & t \text{ odd}, \end{cases} \pmod 2.$$

A short calculation shows that

$$\begin{aligned} \varepsilon^{\mathrm{o}}_{\lambda;(n+1)t} &= \varepsilon^{\mathrm{o}}_{\lambda;nt} + \sum_{r=1}^{\lfloor (t-1)/2\rfloor} m_r(\lambda;nt) + \begin{cases} n + \frac{t}{2} + \operatorname{rk}(t\text{-core}(\lambda)) & t \text{ even},\\ \frac{t-1}{2} & t \text{ odd}, \end{cases}\\ &\equiv \varepsilon^{\mathrm{o}}_{\lambda;nt} + \begin{cases} \frac{(n+1)t}{2} & t \text{ even},\\ \frac{t-1}{2} + \sum_{r=1}^{(t-1)/2} m_r(\lambda;nt) & t \text{ odd} \end{cases} \pmod 2. \end{aligned}$$

where the last equality uses (16). The remaining two cases follow similarly. □

We conclude this section with a small lemma relating the indices in the Jacobi–Trudi determinants with partition quotients.

Lemma 4.9. *Let λ, μ be partitions of length at most nt and assume that for $0 \leqslant r, s \leqslant t-1$ we have $\lambda_i - i \equiv r \pmod t$, $\mu_j - j \equiv s \pmod t$ for $1 \leqslant i, j \leqslant nt$. If $r - s + z \equiv 0 \pmod t$ for some $z \in \mathbb{Z}$, then*

$$\frac{\lambda_i - \mu_j + j - i + z}{t} = \lambda_k^{(r)} - \mu_\ell^{(s)} - k + \ell + m_r(\lambda;nt) - m_s(\mu;nt) + (r-s+z)/t,$$

for some k, ℓ such that $1 \leqslant k \leqslant m_r(\lambda;nt)$ and $1 \leqslant \ell \leqslant m_s(\mu;nt)$. Alternatively, if $r+s+z \equiv 0 \pmod t$ then

$$\frac{\lambda_i + \mu_j - i - j + z}{t} = \lambda_k^{(r)} + \mu_\ell^{(s)} - k - \ell - 2n + 1 + m_r(\lambda;nt) + m_s(\mu;nt) + (r+s+z)/t$$

for some k, ℓ such that $1 \leqslant k \leqslant m_r(\lambda;nt)$ and $1 \leqslant \ell \leqslant m_s(\mu;nt)$.

Proof. We first write $\lambda_i + nt - i = \xi_k^{(r)} t + r$ and $\mu_j + nt - j = \pi_\ell^{(s)} t + s$ for $1 \leqslant k \leqslant m_r(\lambda;nt)$ and $1 \leqslant \ell \leqslant m_s(\mu;nt)$. Then

$$\begin{aligned} \frac{\lambda_i - \mu_j - i + j + z}{t} &= \xi_k^{(r)} + \pi_\ell^{(s)} + (r-s+z)/t\\ &= \lambda_k^{(r)} + \mu_\ell^{(s)} - k + \ell + m_r(\lambda;nt) - m_s(\mu;nt) + (r-s+z)/t, \end{aligned}$$

by the definition of the t-quotient. The second claim is analogous. □

5. Proofs of theorems

In this section we provide proofs of Theorems 3.2, 3.3 and 3.4. Since our proof strategy follows that of Macdonald's proof of the skew Schur case [30, p. 91] (Theorem 3.1 above), we reproduce this proof in detail as preparation for what remains. We also give a detailed example in the orthogonal case in Section 5.2 to further elucidate the structure of the remaining proofs.

5.1. Proof of Theorem 3.1. Let n be a nonnegative integer and $\mu \subseteq \lambda$ be a pair of partitions such that $l(\lambda) \leqslant nt$. Consider the Jacobi–Trudi determinant

$$s_{\lambda/\mu} = \det_{1 \leqslant i,j \leqslant nt} (h_{\lambda_i - \mu_j - i + j}).$$

Before applying the map φ_t, we rearrange the rows and columns of this determinant by the permutations $w_t(\lambda; nt)$ and $w_t(\mu; nt)$ respectively. By Lemma 4.5 this introduces a sign of $\mathrm{sgn}_t(\lambda/\mu)$. The rows and columns are now arranged in such a way that the residue classes of $\lambda_i - i$ and $\mu_j - j$ are grouped in ascending order, and the values within each class are decreasing. From this vantage point it is easy to apply the map φ_t since $\varphi_t h_{\lambda_i - \mu_j - i + j}$ vanishes unless $\lambda_i - i \equiv \mu_j - j \pmod{t}$. Therefore, $\varphi_t s_{\lambda/\mu}$ has a block-diagonal structure, with each block having size $m_r(\lambda; nt) \times m_r(\mu; nt)$ for $0 \leqslant r \leqslant t-1$. We conclude that $\varphi_t s_{\lambda/\mu} = 0$ unless $m_r(\lambda; nt) = m_r(\mu; nt)$ for all $0 \leqslant r \leqslant t-1$. Assuming this is the case, then the entries of the of the minor corresponding to the residue class r are given by Lemma 4.9, and are

$$h_{(\lambda_i - \mu_j - i + j)/t} = h_{\lambda_k^{(r)} - \mu_\ell^{(r)} - k + \ell}$$

for some k and ℓ with $1 \leqslant k, \ell \leqslant n$. Note that the rows and columns are in the desired order (i.e. in each $n \times n$ minor the indices increase from 1 to n) thanks to the permutations we applied at the beginning of the proof. We have therefore shown that if $m_r(\lambda; nt) = m_r(\mu; nt)$ for all $0 \leqslant r \leqslant t-1$, then

$$\varphi_t s_{\lambda/\mu} = \mathrm{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}.$$

Now, if $\mu^{(r)} \not\subseteq \lambda^{(r)}$ for any r such that $0 \leqslant r \leqslant t-1$ this expression will give zero, from which we conclude, by Lemma 2.1, that $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is t-tileable.

5.2. An example. The structure of the remaining proofs is best outlined through a detailed example. To this end, let $t = 4$ and $\lambda = (12, 12, 12, 8, 8, 8, 7, 7, 3, 3, 2)$. We therefore have that 4-core$(\lambda) = (4, 1, 1)$, which is clearly orthogonal, and

$$\big(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big) = \big((2,2), (4,1), (3,2,1), (2,1,1)\big).$$

Now choose $n = 3$, so that $nt = 12 \geqslant l(\lambda)$. Using the definition of o_λ as a Jacobi–Trudi-type determinant (7) we immediately see that

$$\varphi_4 \mathrm{o}_\lambda = \begin{vmatrix}
h_3 & \cdot & -h_2 & \cdot & h_4 & \cdot & -h_1 & \cdot & h_5 & \cdot & -1 & \cdot \\
\cdot & h_3 - h_2 & \cdot & \cdot & \cdot & h_4 - h_1 & \cdot & \cdot & \cdot & h_5 - 1 & \cdot & \cdot \\
-h_2 & \cdot & h_3 & \cdot & -h_1 & \cdot & h_4 & \cdot & -1 & \cdot & h_5 & \cdot \\
\cdot & \cdot & \cdot & h_2 - 1 & \cdot & \cdot & \cdot & h_3 & \cdot & \cdot & \cdot & h_4 \\
h_1 & \cdot & -1 & \cdot & h_2 & \cdot & \cdot & \cdot & h_3 & \cdot & \cdot & \cdot \\
\cdot & h_1 - 1 & \cdot & \cdot & \cdot & h_2 & \cdot & \cdot & \cdot & h_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & h_1 & \cdot & \cdot & \cdot & h_2 & \cdot & \cdot & \cdot & h_3 \\
1 & \cdot & \cdot & \cdot & h_1 & \cdot & \cdot & \cdot & h_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & h_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & h_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{vmatrix},$$

where we write $\cdot$ in place of 0 to avoid clutter. The next step is to permute the rows and columns of the matrix according to the permutations $w_4(\lambda; 12)$ and $w_4(0; 12)$, respectively. In this case, the first permutation is odd and the second even, so we are left with

$$\varphi_4 \mathrm{o}_\lambda = - \begin{vmatrix} h_2-1 & h_3 & h_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_1 & h_2 & h_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & h_3 & h_4 & h_5 & \cdot & \cdot & \cdot & -h_2 & -h_1 & -1 \\ \cdot & \cdot & \cdot & \cdot & 1 & h_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & h_3-h_2 & h_4-h_1 & h_5-1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & h_1-1 & h_2 & h_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & h_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -h_2 & -h_1 & -1 & \cdot & \cdot & \cdot & h_3 & h_4 & h_5 \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & h_1 & h_2 & h_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & h_1 & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix}.$$

The top-left 3×3 minor and central 3×3 minor occupying rows 6–8 and columns 7–9 are clearly equal to $\mathrm{o}_{(2,2)}$ and $\mathrm{so}^-_{(3,2,1)}$, respectively. One way to isolate the copy of $\mathrm{so}^-_{(3,2,1)}$ is to push it so that it is the bottom-right 3×3 submatrix, while preserving the order of the other rows and columns. In this case such a procedure will introduce a sign of -1. Putting this together, we have shown that

$$\varphi_4 \mathrm{o}_\lambda = \mathrm{o}_{(2,2)} \mathrm{so}^-_{(3,2,1)} \begin{vmatrix} h_3 & h_4 & h_5 & -h_2 & -h_1 & -1 \\ \cdot & 1 & h_1 & \cdot & \cdot & \cdot \\ -h_2 & -h_1 & -1 & h_3 & h_4 & h_5 \\ -1 & \cdot & \cdot & h_1 & h_2 & h_3 \\ \cdot & \cdot & \cdot & 1 & h_1 & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix}.$$

Our goal is to show that this final unidentified determinant is equal to $\mathrm{rs}_{(4,1),(2,1,1)}$. Clearly the extra signs can be cleared by multiplying the first two rows and first three columns by -1 each, generating an overall sign of -1. Then one need only push the first column past the second and third, which does not change the sign, and the resulting determinant is precisely a copy of $\mathrm{rs}_{(4,1),(2,1,1)}$. Thus,

$$\varphi_4 \mathrm{o}_\lambda = -\mathrm{o}_{(2,2)} \mathrm{so}^-_{(3,2,1)} \mathrm{rs}_{(4,1),(2,1,1)}.$$

Note that $(-1)^{\varepsilon^{\mathrm{o}}_{\lambda;12}} = 1$ so the overall sign clearly agrees with Theorem 3.2. In the next sections we show that, with a little extra work, this argument also works in general for the universal characters $\mathrm{o}_\lambda, \mathrm{sp}_\lambda$ and so_λ.

5.3. Proof of Theorem 3.2. Let λ be a partition such that $l(\lambda) \leqslant nt$ and consider the definition (7) of o_λ

$$\mathrm{o}_\lambda = \det_{1 \leqslant i,j \leqslant nt} \big(h_{\lambda_i - i + j} - h_{\lambda_i - i - j} \big).$$

We permute the rows and columns by $w_t(\lambda; nt)$ and $w_t(0; nt)$ respectively, which introduces a sign of

$$(-1)^{\binom{n+1}{2}\binom{t}{2}} \operatorname{sgn}(w_t(\lambda; nt)). \tag{19}$$

The modular behaviour of the indices of each row is now known. There are three possibilities for the entries of $\varphi_t \mathrm{o}_\lambda$: both h's may survive, one h may survive, or the entry is necessarily zero. For both to survive, we see that $h_{\lambda_i - i + j}$ and $h_{\lambda_i - i - j}$ are nonzero under φ_t if and only if $\lambda_i - i \equiv -j \equiv 0 \pmod t$ or, if t is even, $\lambda_i - i \equiv -j \equiv t/2 \pmod t$. In the first instance, by Lemma 4.9,

$$\varphi_t \big(h_{\lambda_i - i + j} - h_{\lambda_i - i - j} \big) = h_{\lambda^{(0)}_k - k + \ell + m_0(\lambda; nt) - n} - h_{\lambda^{(0)}_k - k - \ell + m_0(\lambda; nt) - n},$$

where $1 \leqslant k \leqslant m_0(\lambda; nt)$ and $1 \leqslant \ell \leqslant n$. Moreover, all other entries in the first $m_0(\lambda; nt)$ rows and n columns are zero. If t is even then we also find a submatrix of size $m_{t/2}(\lambda; nt) \times n$ in the rows $1 + \sum_{r=0}^{(t-2)/2} m_r(\lambda; nt)$ to $\sum_{r=0}^{t/2} m_r(\lambda; nt)$ and columns $1 + nt/2$ to $n(t+2)/2$. The entries of this submatrix are

$$\varphi_t\big(h_{\lambda_i - i + j} - h_{\lambda_i - i - j}\big) = h_{\lambda_k^{(t/2)} - k + \ell + m_{t/2}(\lambda; nt) - n} - h_{\lambda_k^{(t/2)} - k - \ell + m_{t/2}(\lambda; nt) - n + 1},$$

where $1 \leqslant k \leqslant m_{t/2}(\lambda; nt)$ and $1 \leqslant \ell \leqslant n$. Again, all other entries in these rows and columns are necessarily zero under φ_t. Given a row corresponding to the residue class r where $1 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$, there are two possibilities for the entry to potentially survive: the column corresponds to the residue class r or $t - r$. Again, by Lemma 4.9,

$$\varphi_t\big(h_{\lambda_i - i + j} - h_{\lambda_i - i - j}\big) = \begin{cases} h_{\lambda_k^{(r)} - k + \ell + m_r(\lambda; nt) - n} & \text{if } j \equiv -r \pmod{t}, \\ -h_{\lambda_k^{(r)} - k - \ell + m_r(\lambda; nt) - n + 1} & \text{if } j \equiv r \pmod{t}. \end{cases}$$

The set of indices of the complete homogeneous symmetric functions in such a row are

(20a)

$$\begin{aligned} &\big\{\lambda_k^{(r)} - k - \ell + m_r(\lambda, nt) + 1 \mid 1 \leqslant \ell \leqslant 2n\big\} \\ &= \big\{\lambda_k^{(r)} - k + \ell \mid 1 \leqslant \ell \leqslant m_r(\lambda, nt)\big\} \sqcup \big\{\lambda_k^{(r)} - k - \ell + 1 \mid 1 \leqslant \ell \leqslant m_{t-r}(\lambda, nt)\big\}. \end{aligned}$$

If we look at the complementary row corresponding to $t - r$, then a similar computation shows that the indices are

(20b)

$$\begin{aligned} &\big\{\lambda_k^{(t-r)} - k - \ell + m_{t-r} + 1 \mid 1 \leqslant \ell \leqslant 2n\big\} \\ &= \big\{\lambda_k^{(t-r)} - k + \ell \mid 1 \leqslant \ell \leqslant m_{t-r}(\lambda, nt)\big\} \sqcup \big\{\lambda_k^{(t-r)} - k - \ell + 1 \mid 1 \leqslant \ell \leqslant m_r(\lambda, nt)\big\}. \end{aligned}$$

We have now identified the entries which do not necessarily vanish under φ_t. These can be rearranged into a block-diagonal matrix. If t is even, we move the submatrix corresponding to $t/2$ to the bottom-right $m_{t-1}(\lambda; nt)$ rows and n columns, which picks up a sign of

$$(-1)^{m_{t/2}(\lambda; nt) \sum_{r=(t+2)/2}^{t-1} m_r(\lambda; nt) + n^2 (t-2)/2}.$$

We then group the rows and columns corresponding to the residue classes r and $t - r$ together with $0 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$ increasing. The determinant is now block-diagonal and the blocks have dimension $m_0(\lambda; nt) \times n$, $(m_r(\lambda; nt) + m_{t-r}(\lambda; nt)) \times 2n$ for $1 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$ and, if t is even, $m_{t/2}(\lambda; nt) \times n$. Since the determinant of a block-diagonal matrix vanishes if one of the blocks is not a square, we can therefore conclude that $\varphi_t \mathrm{o}_\lambda$ vanishes unless the conditions (15) with $z = -1$ hold in Lemma 4.3, i.e. unless t-core(λ) is orthogonal. In this case the top-left $n \times n$ minor is equal to $\mathrm{o}_{\lambda^{(0)}}$ and if t is even the bottom-right minor corresponds to $\mathrm{so}^-_{\lambda^{(t/2)}}$. Note that in this case the grouping of the $2n \times 2n$ minors does not change the sign of the determinant since each row and column is pushed past an even number of rows or columns. For each $1 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$ these final minors are of the form

$$\begin{pmatrix} h_{\lambda_1^{(r)} + m_r - n} & \cdots & h_{\lambda_1^{(r)} + m_r - 1} & -h_{\lambda_1^{(r)} + m_r - n - 1} & \cdots & -h_{\lambda_1^{(r)} + m_r - 2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{\lambda_{m_r}^{(r)} + 1 - n} & \cdots & h_{\lambda_{m_r}^{(r)}} & -h_{\lambda_{m_r}^{(r)} - n} & \cdots & -h_{\lambda_{m_r}^{(r)} - 2n + 1} \\ -h_{\lambda_1^{(t-r)} + m_{t-r} - n - 1} & \cdots & -h_{\lambda_1^{(t-r)} + m_{t-r} - 2n} & h_{\lambda_1^{(t-r)} + m_{t-r} - n} & \cdots & h_{\lambda_1^{(t-r)} + m_{t-r} - 1} \\ \vdots & & \vdots & \vdots & & \vdots \\ -h_{\lambda_{m_{t-r}}^{(t-r)} - n} & \cdots & -h_{\lambda_{m_{t-r}}^{(t-r)} - 2n + 1} & h_{\lambda_{m_{t-r}}^{(t-r)} + 1 - n} & \cdots & h_{\lambda_{m_{t-r}}^{(t-r)}} \end{pmatrix},$$

where we write $m_r = m_r(\lambda; nt)$. Clearing the negatives in this minor produces the sign $(-1)^{m_r(\lambda;nt)+n}$. If $m_r(\lambda; nt) = m_{t-r}(\lambda; nt) = n$ then we are done. If $m_r(\lambda; nt) > n$ then we need to move the columns $n+1$ to $m_r(\lambda; nt)$ so they are the first $m_r(\lambda; nt)-n$ columns, and then reverse the order. This gives a sign of

$$(-1)^{n(m_r(\lambda;nt)-n)+\binom{m_r(\lambda;nt)-n}{2}} = (-1)^{\binom{m_r(\lambda;nt)}{2}-\binom{n}{2}}.$$

If $m_r(\lambda; nt) < n$ then we need to push the $m_{t-r}(\lambda; nt) - n$ missing rows past the $n - m_r$ rows to their right and then reverse again, giving the same sign

$$(-1)^{(m_{t-r}(\lambda;nt)-n)m_r(\lambda;nt)+\binom{m_{t-r}(\lambda;nt)-n}{2}} = (-1)^{\binom{m_r(\lambda;nt)}{2}-\binom{n}{2}},$$

since $m_{t-r}(\lambda; nt) - n = n - m_r(\lambda; nt)$. In each of the three cases the resulting determinant is equal to $\mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r)}}$. Collecting all of the above determinant manipulations, the value of $\varepsilon^{\mathrm{o}}_{\lambda;nt}$ is

$$\frac{(n+1)nt(t-1)}{4} + \sum_{r=1}^{\lfloor (t-1)/2\rfloor} \left(\binom{m_r+1}{2} + \binom{n+1}{2}\right) + \begin{cases} n\sum_{r=1}^{(t-2)/2} m_{t-r} + \frac{n^2(t-2)}{2} & t \text{ even},\\ 0 & t \text{ odd}.\end{cases}$$

To see that this agrees with the sign of Ayyer and Kumari, we use (16) together with the fact that for odd t, $(t+1)(t-1)n(n+1)/4$ is even and for even t, $n(n+1)(t^2-2)/4$ has the same parity as $n(n+1)/2$. The above exponent therefore has the same parity as

$$\varepsilon^{\mathrm{o}}_{\lambda;nt} = \sum_{r=\lfloor (t+2)/2\rfloor}^{t-1} \binom{m_r(\lambda;nt)+1}{2} + \mathrm{rk}(t\text{-core}(\lambda)) + \begin{cases} \binom{n+1}{2} + n\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ even},\\ 0 & t \text{ odd}.\end{cases}$$

This completes the proof.

5.4. Proof of Theorem 3.3. It is of course possible to prove Theorem 3.3 by direct manipulation of the h Jacobi–Trudi-type formula for sp_λ (8). However, it will be more insightful to begin with the e Jacobi–Trudi-type formula (9)

$$\mathrm{sp}_\lambda = \det_{1\leqslant i,j\leqslant nt} \left(e_{\lambda'_i-i+j} - e_{\lambda'_i-i-j}\right),$$

where we assume that n is an integer such that $nt \geqslant \lambda_1$. We further assume that $nt \geqslant l(\lambda)$, since, in the end, our sign will be independent of n. The values of $\varphi_t h_r$ and $\varphi_t e_r$ differ by a sign of $(-1)^{(t-1)r/t}$, and the indices of the e's in this formula are the same as the h's in the formula for $\mathrm{o}_{\lambda'}$ ((7) with $\lambda \mapsto \lambda'$), so we can simply replace each h by a signed e in the previous proof. Moreover, by Corollary 4.2, we know that the t-quotient of λ' is simply the reverse of the t-quotient of λ. We can therefore already claim that $\varphi_t\mathrm{sp}_\lambda$ vanishes unless t-core(λ) is symplectic, in which case

$$\varphi_t\mathrm{sp}_\lambda = (-1)^\delta \operatorname{sgn}(w_t(\lambda'; nt))\mathrm{sp}_{\lambda^{(t-1)}} \prod_{i=0}^{\lfloor (t-3)/2\rfloor} \mathrm{rs}_{\lambda^{(i)},\lambda^{(t-i-2)}} \times \begin{cases} \mathrm{so}_{\lambda^{((t-2)/2)}} & t \text{ even},\\ 1 & t \text{ odd},\end{cases}$$

where

$$\delta = (t-1)\sum_{r=0}^{t-1}|\lambda^{(r)}| + \sum_{r=\lfloor (t+1)/2\rfloor}^{t-1} \binom{m_r(\lambda'; nt)}{2} + \mathrm{rk}(t\text{-core}(\lambda)) + \begin{cases} \binom{n+1}{2} + n\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ even},\\ 0 & t \text{ odd}.\end{cases}$$

All that remains now is to show that this sign agrees with that of Theorem 3.3. By a combination of Lemmas 4.6 and 4.7 we may replace $\operatorname{sgn}(w_t(\lambda';nt))$ by $\operatorname{sgn}(w_t(\lambda;nt))$, which cancels $\operatorname{rk}(t\text{-core}(\lambda)) + (t-1)\sum_{r=0}^{t-1}|\lambda^{(r)}|$ in δ. If we call this new exponent δ', then we also have by Lemma 4.1 that $m_r(\lambda';nt) = m_{r-1}(\lambda;nt)$ for $\lfloor (t+1)/2\rfloor \leqslant r \leqslant t-1$, which implies $\delta' = \varepsilon^{\mathrm{sp}}_{\lambda;nt}$.

5.5. Proof of Theorem 3.4. The final proof closely follows the first. Let λ be a partition of length at most nt and consider

$$\mathrm{so}_\lambda = \det_{1\leqslant i,j\leqslant nt}\big(h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1}\big).$$

As before we apply the permutations $w_t(\lambda;nt)$ and $w_t(0;nt)$ to the rows and columns of this determinant, introducing the sign (19). Unlike before, there is only one case in which both h's may survive. If t is odd and $\lambda_i - i \equiv -j \equiv (t-1)/2 \pmod t$ then we have

$$\begin{aligned}\varphi_t\big(&h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1}\big)\\ &= h_{\lambda_k^{((t-1)/2)}-k+\ell+m_{(t-1)/2}(\lambda;nt)-n} + h_{\lambda_k^{((t-1)/2)}-k-\ell+1+m_{(t-1)/2}(\lambda;nt)-n}.\end{aligned}$$

where $1\leqslant k\leqslant m_{(t-1)/2}(\lambda;nt)$ and $1\leqslant \ell\leqslant n$. These entries lie in the rows $1+\sum_{r=0}^{(t-3)/2} m_r(\lambda;nt)$ to $\sum_{r=0}^{(t-1)/2} m_r(\lambda;nt)$ and columns $1+n(t-1)/2$ to $n(t+1)/2$, and outside of their intersection, all other entries in these rows and columns are zero. Now consider a row corresponding to the residue class r for $0\leqslant r\leqslant \lfloor (t-2)/2\rfloor$. Then the column must fall into the residue class r or $t-r-1$ in order for the entry to not necessarily vanish. In this case we now have

$$\varphi_t\big(h_{\lambda_i-i+j} + h_{\lambda_i-i-j+1}\big) = \begin{cases} h_{\lambda_k^{(r)}-k+\ell_1+m_r(\lambda;nt)-n} & \text{if } j\equiv -r \pmod t,\\ h_{\lambda_k^{(r)}-k-\ell_1+m_r(\lambda;nt)-n+1} & \text{if } j\equiv r+1 \pmod t.\end{cases}$$

Again, a similar computation holds for the row corresponding to $t-r-1$, and the sets of indices agree with (20) but with $t-r\mapsto t-r-1$ in (20b). If t is odd we move the central submatrix corresponding to $(t-1)/2$ to the top-left, picking up a sign of

$$(-1)^{m_{(t-1)/2}(\lambda;nt)\sum_{r=0}^{(t-3)/2} m_r(\lambda;nt)+n^2(t-1)/2}.$$

The grouping and rearrangement of the remaining minors is the same as in the first proof above. We only remark that the result is the determinant of a block-diagonal matrix with blocks of sizes $(m_r(\lambda;nt)+m_{t-r-1}(\lambda;nt))\times 2n$ for $0\leqslant r\leqslant\lfloor (t-2)/2\rfloor$ plus one of size $m_{(t-1)/2}(\lambda;nt)\times n$ if t is odd. Thus the determinant vanishes unless (15) holds with $z=0$, i.e. unless t-core(λ) is self-conjugate. Accounting for the sign of $(-1)^{\binom{m_r(\lambda;nt)}{2}+\binom{n}{2}}$ from reordering the columns in the copies of $\mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-1)}}$, the exponent $\varepsilon^{\mathrm{so}}_{\lambda;nt}$ has the value

$$\frac{(n+1)nt(t-1)}{4} + \sum_{r=0}^{\lfloor (t-2)/2\rfloor}\left(\binom{m_r}{2}+\binom{n}{2}\right) + \begin{cases}0 & t \text{ even},\\ n\sum_{r=0}^{(t-3)/2} m_r + n^2(t-1)/2 & t\text{ odd}.\end{cases}$$

By (18) this has the same parity as

$$\varepsilon^{\mathrm{so}}_{\lambda;nt} = \sum_{r=\lfloor (t+1)/2\rfloor}^{t-1}\binom{m_r(\lambda;nt)+1}{2} + \begin{cases}0 & t\text{ even},\\ n\operatorname{rk}(t\text{-core}(\lambda)) & t\text{ odd}.\end{cases}$$

5.6. Proof of Proposition 3.5. To close out this section, we sketch the proof of Proposition 3.5. In the Schur case, by Corollary 4.2 and the fact that λ/μ is t-tileable if and only if λ'/μ' is, we already have $\omega\varphi_t s_{\lambda/\mu} = \pm\varphi_t\omega s_{\lambda/\mu}$. The precise difference in sign is then provided by Lemma 4.6. Again using Corollary 4.2, we have

$$\begin{aligned}\omega\varphi_t \mathrm{so}_\lambda &= (-1)^{\varepsilon^{\mathrm{so}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda;nt)) \prod_{r=0}^{\lfloor (t-2)/2\rfloor} \mathrm{rs}_{(\lambda^{(r)})',(\lambda^{(t-r-1)})'} \times \begin{cases} 1 & t \text{ even},\\ \mathrm{so}_{(\lambda^{((t-1)/2)})'} & t \text{ odd},\end{cases}\\ &= (-1)^{\varepsilon^{\mathrm{so}}_{\lambda;nt}+\varepsilon^{\mathrm{so}}_{\lambda';nt}} \operatorname{sgn}(w_t(\lambda;nt)) \operatorname{sgn}(w_t(\lambda';nt))\varphi_t \mathrm{so}_{\lambda'},\end{aligned}$$

where n should be large enough so that λ is contained in an $nt \times nt$ box. Combining Lemmas 4.5 and 4.6 shows that, in this case,

$$\operatorname{sgn}(w_t(\lambda;nt)) \operatorname{sgn}(w_t(\lambda';nt)) = (-1)^{(t-1)(|\lambda^{(0)}|+\cdots+|\lambda^{(t-1)}|)}.$$

Moreover, $\varepsilon^{\mathrm{so}}_{\lambda;nt} = \varepsilon^{\mathrm{so}}_{\lambda';nt}$, so that the total sign agrees with the claim.

6. Other factorisations

6.1. Littlewood-type factorisations. In [27, §7.3], Littlewood proves a factorisation slightly more general than the one contained in Theorem 3.1 for μ empty; see also [5, Theorem 2.7]. Here, and below, we let $\lambda^{(r)} = \lambda^{(k)}$ if $k \equiv r \pmod t$.

Theorem 6.1. *Let λ be a partition of length at most $nt+1$ and $X = (x_1,\dots,x_n)$ a set of variables. Then for another variable q,*

$$s_\lambda(X,\zeta X,\dots,\zeta^{t-1}X,q) = 0$$

unless t-core$(\lambda) = (c)$ for some $0 \leqslant c \leqslant t-1$, in which case

$$s_\lambda(X,\zeta X,\dots,\zeta^{t-1}X,q) = \operatorname{sgn}_t(\lambda/(c)) q^c s_{\lambda^{(c-1)}}(X^t,q^t) \prod_{\substack{r=0\\ r\neq c-1}}^{t-1} s_{\lambda^{(r)}}(X^t).$$

This theorem can also be placed in our framework, however in a somewhat less elegant manner than our other results. The operator $\varphi_t^q : \Lambda \longrightarrow \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ which gives the above may be defined by

$$\varphi_t^q h_{at+b} := q^b \sum_{k\geqslant 0} q^{kt} h_{a-k} = \sum_{k\geqslant 0} q^k \varphi_t h_{at+b-k}.$$

Note that the sums are finite since h_r vanishes for negative r, and that for $q=0$ this reduces to the operator φ_t from the earlier sections. Alternatively, since

$$h_r(X,q) = \sum_{k\geqslant 0} q^k h_{r-k}(X),$$

the image of φ_t^q acting on any symmetric function f is the same as the image of φ_t acting on $f(X,q)$, where φ_t acts only on the X variables. Using φ_t^q, Littlewood's above theorem may be phrased as follows. After the statement we provide a short proof which relies only on Theorem 3.1.

Proposition 6.2. *We have that $\varphi_t^q s_\lambda = 0$ unless t-core$(\lambda) = (c)$ for some c such that $0 \leqslant c \leqslant t-1$, in which case*

$$\varphi_t^q s_\lambda = \operatorname{sgn}_t(\lambda/(c)) q^c \prod_{\substack{r=0\\ r\neq c-1}}^{t-1} s_{\lambda^{(r)}} \sum_{k\geqslant 0} q^{kt} s_{\lambda^{(c-1)}/(k)}.$$

Proof. The first observation is that

$$\varphi_t^q s_\lambda = \sum_{k\geqslant 0} q^k \varphi_t s_{\lambda/(k)},$$

which is a simple consequence of the branching rule for Schur functions [30, p. 72]. In the case that $l(t\text{-core}(\lambda)) > 1$ then each term in the sum on the right-hand side vanishes by Theorem 3.1 as the t-cores of the inner and outer shape can never be equal. Now assume $t\text{-core}(\lambda) = (c)$ for some $0 \leqslant c \leqslant t-1$, which is a complete set of t-cores with length one. Then the nonzero terms in the sum on the right-hand side are those for which k is of the form $\ell t + c$ with $\ell \geqslant 0$ and $\ell t + c \leqslant \lambda_1$. Therefore

$$\varphi_t^q s_\lambda = q^c \sum_{\ell\geqslant 0} q^{\ell t} \varphi_t s_{\lambda/(\ell t+c)} = q^c \sum_{\ell\geqslant 0} \operatorname{sgn}_t(\lambda/(\ell t + c)) q^{\ell t} s_{\lambda^{(c-1)}/(\ell)} \prod_{\substack{r=0\\ r\neq c-1}}^{t-1} s_{\lambda^{(r)}},$$

again by Theorem 3.1. By our convention we always compute the t-quotient using a beta set with number of elements a multiple of t. This means that the single row $(\ell t+c)$ has one non-empty element in its t-quotient, $\lambda^{(c-1)}$. Moreover, since the partitions $(\ell t + c)$ all differ by a ribbon of height zero, the sign of each term in the sum is the same and equal to $\operatorname{sgn}_t(\lambda/(c))$. Putting all of this together, we arrive at

$$\varphi_t^q s_\lambda = \operatorname{sgn}_t(\lambda/(c)) q^c \prod_{\substack{r=0\\ r\neq c-1}}^{t-1} s_{\lambda^{(r)}} \sum_{\ell\geqslant 0} q^{\ell t} s_{\lambda^{(c-1)}/(\ell)}. \qquad \square$$

We do not see, at this stage, whether it is possible to extend the previous result to skew Schur functions. If we expand $\varphi_t^q s_{\lambda/\mu}$ as in the proof above, we find that

$$\varphi_t^q s_{\lambda/\mu} = \sum_{\nu\succ\mu} q^{|\nu|-|\mu|} \varphi_t s_{\lambda/\nu},$$

where $\nu \succ \mu$ means that ν/μ is a *horizontal strip*, i.e. $\nu \supseteq \mu$ and ν/μ contains at most one box in each column of its Young diagram. Of course, this implies that $\varphi_t^q s_{\lambda/\mu} = 0$ if there does not exist a ν such that $\nu \succ \mu$ and λ/ν is t-tileable. However, the sum may vanish even if such a ν exists. For example,

$$\begin{aligned}
\varphi_2^q s_{(4,4)/(1)} &= q\varphi_t\big(s_{(4,4)/(2)} + s_{(4,4)/(1,1)}\big) + q^3\varphi_t\big(s_{(4,4)/(4)} + s_{(4,4)/(3,1)}\big)\\
&= q\big(s_{(2)}s_{(2)/(1)} - s_{(2)}s_{(2)/(1)}\big) + q^3\big(s_{(2)} - s_{(2)}\big)\\
&= 0.
\end{aligned}$$

In a similar direction Pfannerer [35, Theorem 4.4] has shown that, if λ has empty t-core and $m = \ell t+k$ is any integer, then the Schur function $s_\lambda(1, \zeta, \dots, \zeta^{m-1})$ always factors as a product of Schur functions with variables all one indexed by the t-quotient of λ. When m is a multiple of t this becomes a special case of Littlewood's theorem (Theorem 3.1 with μ empty) noted by Reiner, Stanton and White [37, Theorem 4.3]. Pfannerer's result has subsequently been generalised by Kumari [21, Theorem 2.2], in addition to analogues of Theorem 6.1 for other classical group characters. It is an open problem to see how these factorisations fit into our story.

6.2. Factorisations of supersymmetric Schur functions. Recently, Kumari has given a version of Theorem 3.1 for the so-called *skew hook Schur functions* (or *supersymmetric skew Schur functions*) [22, Theorem 3.2]. For two independent sets of variables (*alphabets*), we denote their plethystic difference by $X - Y$; see, e.g., [14, 24] for the necessary background on plethystic notation. We also note that for an alphabet X, we let εX be the alphabet with all variables negated. The *complete homogeneous supersymmetric function* used in [22] may be defined as

$$\sum_{j=0}^{r} h_j(X)e_{r-j}(Y) = h_r[X-\varepsilon Y].$$

The hook Schur function is then the Jacobi–Trudi determinant of these functions, so that

$$s_{\lambda/\mu}[X-\varepsilon Y] = \det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-\mu_j-i+j}[X-\varepsilon Y]\big).$$

From this, it follows readily that Kumari's factorisation for the hook Schur functions is contained in Theorem 3.1 above at the alphabet $X-\varepsilon Y$.

6.3. Factorisations of $\mathrm{rs}_{\lambda,\mu}$. To close, we point out that the universal character $\mathrm{rs}_{\lambda,\mu}$ can be used to lift some factorisation results, discovered by Ciucu and Krattenthaler [8, Theorems 3.1–3.2] and subsequently generalised by Ayyer and Behrend [3, Theorems 1–2], to the universal character level. In the next result we write $\lambda+1^n = (\lambda_1+1,\dots,\lambda_n+1)$ where $n\geqslant l(\lambda)$.

Theorem 6.3. *For λ a partition of length at most n, there holds*

$$\mathrm{rs}_{\lambda,\lambda} = \mathrm{so}_\lambda \mathrm{so}^-_\lambda, \tag{21a}$$

and

$$\mathrm{rs}_{\lambda+1^n,\lambda} = \mathrm{o}_{\lambda+1^n}\mathrm{sp}_\lambda. \tag{21b}$$

Moreover, for λ a partition of length at most $n+1$,

$$\begin{aligned}\mathrm{rs}_{(\lambda_1,\dots,\lambda_n),(\lambda_2,\dots,\lambda_{n+1})}&+\mathrm{rs}_{(\lambda_1-1,\dots,\lambda_{n+1}-1),(\lambda_2+1,\dots,\lambda_n+1)}\\ &= \mathrm{sp}_{(\lambda_1,\dots,\lambda_n)}\mathrm{o}_{(\lambda_2,\dots,\lambda_{n+1})},\end{aligned} \tag{22a}$$

and

$$\begin{aligned}\mathrm{rs}_{(\lambda_1+1,\dots,\lambda_n+1),(\lambda_2,\dots,\lambda_{n+1})}&+\mathrm{rs}_{(\lambda_1,\dots,\lambda_{n+1}),(\lambda_2+1,\dots,\lambda_n+1)}\\ &= \mathrm{so}_{(\lambda_1+1,\dots,\lambda_n+1)}\mathrm{so}^-_{(\lambda_2,\dots,\lambda_{n+1})}.\end{aligned} \tag{22b}$$

To get back to the results of Ayyer and Behrend one simply evaluates both sides of each equation at the alphabet $(x_1^{\pm},\dots,x_n^{\pm})$. The precise forms present in [3, Equations (18)–(21)] then follow from (6).(2) As identities for Laurent polynomials the pairs of identities (21) and (22) admit uniform statements. However no such uniform statement will exist for the above generalisation, since this requires characters indexed by half-partitions, which cannot be handled by the universal characters. Ayyer and Fischer [4] have also given skew analogues of the non-universal case of Theorem 6.3. Jacobi–Trudi formulae for the symplectic and orthogonal characters have recently been derived in [1, 17], and so there are candidates for the universal characters for those objects. However, the main obstacle in lifting Ayyer and Fischer's results to the universal level is the lack of a skew analogue of $\mathrm{rs}_{\lambda,\mu}$.

Proof of Theorem 6.3. First up is (21a), which is the simplest of the four. In the determinant

$$\mathrm{rs}_{\lambda,\lambda} = \det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n}\\ \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix},$$

(2)The factor of $(1+\delta_{0,\lambda_{n+1}})$ in [3, Equation (20)] is not present in our generalisation (22a) since the second character vanishes if $\lambda_{n+1}=0$.

add the blocks on the right to the blocks on the left, and then subtract the blocks on the top from the blocks on the bottom, giving

$$\begin{aligned}\mathrm{rs}_{\lambda,\lambda} &= \det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n}\\ 0 & \big(h_{\lambda_i-i+j}-h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix}\\ &= \mathrm{so}_\lambda\mathrm{so}^-_\lambda.\end{aligned}$$

For the second identity (21b),

$$\mathrm{rs}_{\lambda+1^n,\lambda} = \det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+2}\big)_{1\leqslant i,j\leqslant n}\\ \big(h_{\lambda-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix},$$

and we add columns $1,\dots,n-1$ to the columns $n+2,\dots,2n$ and then subtract the bottom two blocks from the top two, resulting in

$$\begin{aligned}&\mathrm{rs}_{\lambda+1^n,\lambda}\\ &\quad= \frac{1}{2}\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j+1}-h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n} & 0\\ \big(h_{\lambda-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+2}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix}\\ &\quad= \mathrm{o}_{\lambda+1^n}\mathrm{sp}_\lambda.\end{aligned}$$

In the third identity, we consider the second determinant in the sum in (22a)

$$\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j-1}\big)_{1\leqslant i,j\leqslant n+1} & \big(h_{\lambda_i-i-j}\big)_{\substack{1\leqslant i\leqslant n+1\\ 1\leqslant j\leqslant n-1}}\\ \big(h_{\lambda_{i+1}-i-j+2}\big)_{\substack{1\leqslant i\leqslant n-1\\ 1\leqslant j\leqslant n+1}} & \big(h_{\lambda_{i+1}-i+j+1}\big)_{1\leqslant i,j\leqslant n-1}\end{pmatrix}.$$

Push the first column so it becomes the $(n+1)$-st, and then push the $(n+1)$-st row to the final row, which picks up a minus sign. The resulting determinant differs from that of $\mathrm{rs}_{(\lambda_1,\dots,\lambda_n),(\lambda_2,\dots,\lambda_{n+1})}$ in only the last row, so we can take the sum of the two, giving

$$\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i\leqslant n}\\ \big(h_{\lambda_{i+1}-i-j+1}\big)_{\substack{1\leqslant i\leqslant n-1\\ 1\leqslant j\leqslant n}} & \big(h_{\lambda_{i+1}-i+j}\big)_{\substack{1\leqslant i\leqslant n-1\\ 1\leqslant j\leqslant n}}\\ (h_{\lambda_{n+1}-n-j+1}-h_{\lambda_{n+1}-n+j-1})_{1\leqslant j\leqslant n} & (h_{\lambda_{n+1}-n+j}-h_{\lambda_{n+1}-n-j})_{1\leqslant j\leqslant n}\end{pmatrix}.$$

In this new determinant, add columns $n+1,\dots,2n-1$ to columns $2,\dots,n$, and then subtract rows $2,\dots,n$ from rows $n+1,\dots,2n-1$, which gives

$$\frac{1}{2}\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+2}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i\leqslant n}\\ 0 & \big(h_{\lambda_{i+1}-i+j}-h_{\lambda_{i+1}-i-j}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix},$$

which equals $\mathrm{sp}_{(\lambda_1,\dots,\lambda_n)}\mathrm{o}_{(\lambda_2,\dots,\lambda_{n+1})}$. The final factorisation (22b) follows almost identically. ☐

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