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Cofibration category of digraphs for path homology

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ABSTRACT We prove that the category of directed graphs and graph maps carries a cofibration category structure in which the weak equivalences are the graph maps inducing isomorphisms on path homology.

1. INTRODUCTION

Homology theories, among other homotopy-theoretic invariants, play an important role in graph theory. Examples of such homology theories of graphs (cf. [5, §2.4]) include the clique homology, which is the homology of the clique complex associated to a graph; CW-homology, i.e. the homology of the graph viewed as a 1-dimensional CW-complex; and cubical homology, which is the homology of the 1-coskeletal cubical set associated to a graph [4]. Path homology, introduced by Grigor’yan, Lin, Muranov, and Yau [15], is yet another such invariant, however it is fundamentally an invariant of *directed* graphs, or digraphs. It is most closely related to magnitude homology [19], as shown recently by Asao [1].

Path homology has seen significant development over the last 10 years. On the foundational side, this includes the development of the corresponding homotopy theory [16], which in turn allows for the statement and proof of the Eilenberg–Steenrod axioms [14]. On the computational side, a Künneth-style theorem was proven in [18]. Finally, these techniques found applications both within mathematics, e.g. a new proof of the classical Sperner Lemma [16, §5], and outside, e.g. in directed network analysis [11, 10].

Since its introduction, path homology has been vastly generalized. First, from digraphs to path complexes [17], which are combinatorial objects similar to, yet more general than, simplicial complexes. In particular, both digraphs and simplicial complexes are canonically examples of path complexes and path homology of path complexes specializes both to path homology of digraphs and to simplicial homology of simplicial complexes. The second generalization [21] was to the category of path sets, a presheaf category similar to that of simplicial sets.

The goal of the present paper is to investigate path homology using tools from abstract homotopy theory, in particular, the framework of cofibration categories. Our main theorem is:

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THEOREM (cf. Theorem 5.1). *The category of directed graphs carries a cofibration category structure in which the weak equivalences are the graph maps inducing isomorphisms on path homology groups.*

(Co)fibration categories were developed by Brown [7] under the name ‘categories of fibrant objects,’ as a framework for studying generalized cohomology theories, but have since found many other applications, e.g. in formal logic [2]. Cofibration categories are a slight weakening of a more common notion of a model category, as developed by Quillen [24]. More precisely, a cofibration category structure on a category \mathcal{C} consists of two classes of morphisms in \mathcal{C} : cofibrations and weak equivalences, subject to some axioms making it possible to speak of and conveniently work with homotopy colimits in \mathcal{C} [27, 22].

To define our cofibration category structure, we build on the development of path homology, especially in papers [14] and [18]. In particular, our definition of cofibration (Definition 3.9) is a strengthening of the ‘no-outgoing-edges condition’ used in [14, §5]. However, as explained in Remark 3.11, this would not be sufficient and hence we require the existence of a projecting decomposition, introduced in [23].

Our work provides additional insight into the structural properties of path homology. For instance, the fact that the class of cofibrations in Definition 3.9 is not the saturation of a small set (Proposition 3.20) suggests that additional axioms will be required, as indicated in [14, Rmk. 5.3], to uniquely determine path homology.

1.1. RELATED WORK. While we are unaware of similar work in the category of digraphs, considerations similar to ours are not without precedent in the category of (undirected) graphs. In [9], a fibration category structure is constructed on the category of simple graphs in which the weak equivalences are the weak homotopy equivalences of discrete homotopy theory [3]. On the other hand, in [13], it is proven that no model category structure exists in the category of undirected graphs with loops in which weak equivalences are the \times -homotopy equivalences of Dochtermann [12] and cofibrations are a subclass of monomorphisms.

1.2. ORGANIZATION. This paper is structured as follows. We begin by recalling the necessary notions related to digraphs, path homology, and cofibration categories in Section 2. In Section 3, we introduce our notion of cofibration of digraphs and study their basic properties. The technical heart of the paper is contained in Section 4, where we prove the excision property, i.e. that relative homology induces isomorphisms on homotopy pushouts. Finally, we assemble all of these results together in Section 5, proving our main theorem.

2. PRELIMINARIES

In this section, we review and establish the necessary background for the results of Sections 3, 4 and 5. We begin by defining (the category of) digraphs and establishing a few facts about colimits therein. We then review the notion of path homology of a digraph, following [15, 16, 17], computing path homology of a few small graphs and referencing our Python script [8] for computations of larger examples. Finally, we review the requisite background on cofibration categories in preparation for our main theorem asserting the existence of a cofibration category of digraphs.

2.1. THE CATEGORY OF DIGRAPHS. We begin by defining the category of directed graphs, or digraphs, as a reflective subcategory of a presheaf category.

DEFINITION 2.1. Define the category \mathbb{G} to be generated by the graph

$$V \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{r} \\ \xrightarrow{t} \end{array} E$$

subject to the relations $rs = rt = \text{id}$.

DEFINITION 2.2. The category of directed multigraphs $\text{Set}^{\mathbb{G}^{\text{op}}}$ is the category of functors $\mathbb{G}^{\text{op}} \rightarrow \text{Set}$.

Explicitly, a directed multigraph X consists of a set X_V of vertices and a set X_E of edges, together with functions (which we denote with a slight abuse of notation):

$$s, t: X_E \rightarrow X_V \quad r: X_V \rightarrow X_E$$

where s and t pick out the source and target vertices of each (directed) edge and $r: X_V \rightarrow X_E$ sends a vertex to a “degenerate” self-edge. A morphism $f: X \rightarrow Y$ of directed multigraphs is a natural transformation, i.e. a pair of functions $(f_V: X_V \rightarrow Y_V, f_E: X_E \rightarrow Y_E)$ which preserve sources, targets, and “degenerate” self-edges. That is, for $e \in X_E$ and $v \in X_V$,

$$s(f_E(e)) = f_V(s(e)) \quad t(f_E(e)) = f_V(t(e)) \quad r(f_V(v)) = f_E(r(v)).$$

We denote an edge of a directed multigraph $e \in X_E$ with source $v \in X_V$ and target $w \in X_V$ by $v \rightarrow w$.

DEFINITION 2.3. A directed graph (or digraph) is a directed multigraph $X: \mathbb{G}^{\text{op}} \rightarrow \text{Set}$ such that the function

$$X_E \xrightarrow{(s,t)} X_V \times X_V$$

is injective; i.e. there is at most one edge $v \rightarrow w$ for any pair of vertices (v, w) .

Let DiGraph denote the full subcategory of $\text{Set}^{\mathbb{G}^{\text{op}}}$ spanned by digraphs. For a morphism $f: X \rightarrow Y$ between digraphs, the function $f_E: X_E \rightarrow Y_E$ is uniquely determined by $f_V: X_V \rightarrow Y_V$. Thus, the data of a digraph map consists of a function $X_V \rightarrow Y_V$ between vertices such that if $v \rightarrow w$ in X then either $f(v) = f(w)$ (as every vertex has a self-edge) or $f(v) \rightarrow f(w)$ in Y .

REMARK 2.4. One may equivalently define a digraph as a set with a reflexive binary relation and a digraph map as a function which preserves this relation.

To fix the notation for specific digraphs used later in the paper, we now discuss several examples of digraphs.

EXAMPLE 2.5. The empty digraph \emptyset is given by the functor $X: \mathbb{G}^{\text{op}} \rightarrow \text{Set}$ with $X_V = X_E = \emptyset$. This is an initial object in DiGraph .

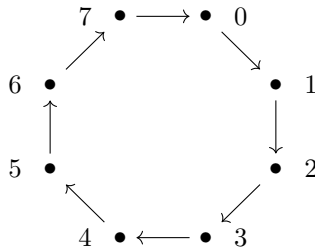
DEFINITION 2.6. For each $n \geq 0$, the digraph I_n has vertices $0, 1, \dots, n$, and a unique edge $i \rightarrow i + 1$ for each $0 \leq i < n$. It can be depicted as

$$\begin{array}{ccccccc} 0 & & 1 & & 2 & & \dots & & n \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \dots & \longrightarrow & \bullet \end{array}$$

In particular, the graph I_0 consists of a single vertex; it is a terminal object in DiGraph . Note that maps $I_0 \rightarrow X$ are in a one-to-one correspondence with vertices of X , i.e. the functor $(-)_V: \text{DiGraph} \rightarrow \text{Set}$ taking a digraph to its set of vertices is representable, represented by I_0 .

Similarly, the functor $(-)_E: \text{DiGraph} \rightarrow \text{Set}$ taking a digraph to its set of edges is representable, represented by I_1 . Indeed, maps $I_1 \rightarrow X$ correspond bijectively to edges of X (including degenerate edges).

DEFINITION 2.7. For each $n \geq 0$, the oriented n -cycle C_n is the digraph with vertices $0, 1, \dots, n - 1$, edges $i \rightarrow i + 1$ for each $0 \leq i < n - 1$, and an edge $n - 1 \rightarrow 0$. For example, the cycle C_8 may be depicted as



DEFINITION 2.8. For each $k \geq 1$, the alternating n -cycle \tilde{C}_{2k} has vertices $0, 1, \dots, 2k - 1$ and edges $i \rightarrow i + 1$ for each $0 \leq i \leq 2k - 1$ even, and $i + 1 \rightarrow i$ for each $0 \leq i \leq 2k - 1$ odd (where vertices are taken modulo $2k$ as needed). For example:

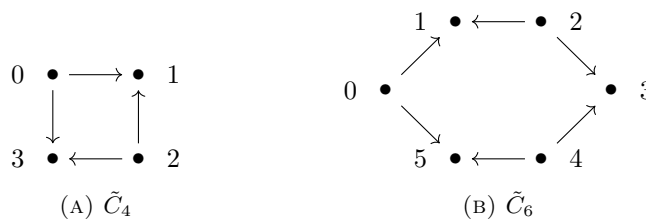


FIGURE 1. Depictions of alternating cycles

DEFINITION 2.9. For each $m, n \geq 0$, the (m, n) -cycle $C_{m,n}$ is the digraph with vertices $0, 1, \dots, m + n - 1$, edges $i \rightarrow i + 1$ for each $0 \leq i < m$, edges $i + 1 \rightarrow i$ for each $m \leq i < m + n - 1$, and an edge $0 \rightarrow m + n - 1$. For example, the cycles $C_{2,1}$, $C_{3,1}$ and $C_{3,2}$ may be depicted:

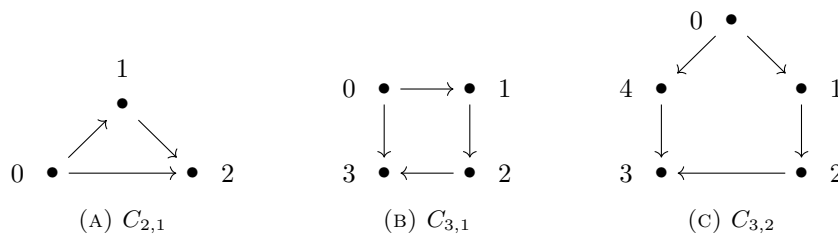


FIGURE 2. Depictions of $C_{m,n}$ cycles

The inclusion $\text{DiGraph} \hookrightarrow \text{Set}^{\text{Gop}}$ admits a left adjoint taking a directed multigraph to the digraph obtained by collapsing all parallel edges. It follows that the category of digraphs admits all small limits and colimits. Moreover, the limits are computed separately on vertices and edges, while the colimits are first computed in the category Set^{Gop} of directed multigraphs and then reflected using the left adjoint.

The following results provide a convenient characterization of pushouts in DiGraph of induced subgraph inclusions.

LEMMA 2.10. Let $A \hookrightarrow X$ denote an inclusion of directed graphs, with A an induced subgraph of X , and let $B \hookrightarrow Y$ denote its pushout along a map $f: A \rightarrow B$, as depicted below.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Then the vertex set of the pushout object Y is given by $Y_V = (X_V \setminus A_V) \sqcup B_V$. Edges of Y are determined as follows.

- For $x, y \in X_V \setminus A_V$, there is an edge $x \rightarrow y$ in Y if and only if there is an edge $x \rightarrow y$ in X .
- For $a, b \in B_V$, there is an edge $a \rightarrow b$ in Y if and only if there is an edge $a \rightarrow b$ in B .
- For $x \in X_V \setminus A_V$ and $a \in B_V$, there is an edge $x \rightarrow a$ if and only if there is an edge $x \rightarrow \bar{a}$ for some $\bar{a} \in A_V$ with $f(\bar{a}) = a$. Similarly, there is an edge $a \rightarrow x$ if and only if there is an edge $\bar{a} \rightarrow x$ for some $\bar{a} \in A_V$ with $f(\bar{a}) = a$.

The map $X \rightarrow Y$ acts as the identity on the complement $X \setminus A$ and restricts to f on A . □

COROLLARY 2.11. In the situation of Lemma 2.10, the map $X \rightarrow Y$ restricts to an isomorphism of complements $X \setminus A \cong Y \setminus B$. Moreover, a vertex $y \in Y_V \setminus B_V$ admits a path to some vertex of B if and only if the vertex $x \in X \setminus A$ corresponding to y under the above isomorphism admits a path to some vertex of A . In this case, the minimum length of a path in Y from y to a vertex of B is equal to the minimum length of a path in X from x to a vertex of A . □

We will also use the following construction:

DEFINITION 2.12. The box product of two graphs X and Y is the graph $X \square Y$ with vertices $X_V \times Y_V$ and an edge $(x, x') \rightarrow (y, y')$ when either of the following conditions holds:

- there is an edge $x \rightarrow y$ in X and $x' = y'$, or
- there is an edge $x' \rightarrow y'$ in Y and $x = y$.

2.2. PATH HOMOLOGY. We now define path homology of digraphs, following [15, 16].

DEFINITION 2.13. Let X be a digraph, and let R be a commutative ring. Define the following R -modules

$$\begin{aligned} K_n(X; R) &= R\{X_V^{n+1}\}, \\ DK_n(X; R) &= R\{(x_0, x_1, \dots, x_n) \in X_V^{n+1} \mid x_i = x_{i+1} \text{ for some } 0 \leq i < n\}, \\ C_n(X; R) &= K_n X / DK_n X. \end{aligned}$$

A generator of $K_n(X; R)$ can be thought of as a path of length n in the complete digraph on the vertices of X . The generators of the submodule $DK_n(X; R)$ can be thought of as *degenerate* paths of length n that contain self-loops $x \rightarrow x$, which are referred to as *non-regular* paths in [15, §2.3]. The quotient $C_n(X; R)$ is then generated by the regular paths, which are paths that do not contain self-loops.

There are differentials on $K_n(X; R)$ given by the usual alternating sum formula:

$$\partial_n(v_0, \dots, v_n) = \sum_{i=0}^n (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_n).$$

These differentials satisfy $\partial^2 = 0$ (i.e. $\partial_{n-1}\partial_n = 0$). Further, ∂_n restricts to a map $DK_n(X; R) \rightarrow DK_{n-1}(X; R)$, and thus passes to the quotient, resulting in a well-defined map:

DEFINITION 2.14. Let $\partial_n: C_n(X; R) \rightarrow C_{n-1}(X; R)$ be the map induced by $\partial_n: K_n(X; R) \rightarrow K_{n-1}(X; R)$.

LEMMA 2.15 ([15, Lemma 2.4]). With the definitions above, $(C_\bullet(X; R), \partial_\bullet)$ is a chain complex. \square

DEFINITION 2.16. Let X be a digraph, and let R be a commutative ring. Define the R -modules:

$$\begin{aligned} \tilde{A}_n(X; R) &= R(\text{DiGraph}(I_n, X)), \\ D\tilde{A}_n(X; R) &= R\{f: I_n \rightarrow X \mid f \text{ factors through a map } I_n \rightarrow I_{n-1}\}, \\ A_n(X; R) &= \tilde{A}_n(X; R)/D\tilde{A}_n(X; R). \end{aligned}$$

In the nomenclature of [15], the elements of $\text{DiGraph}(I_n, X)$ are called *allowed paths*, and consequently $\tilde{A}_n(X; R)$ is the R -module of *allowed paths*. The quotient $A_n(X; R)$ is then the R -module of *allowed regular paths*.

Since $A_n(X; R)$ is a submodule of $C_n(X; R)$, we may restrict $\partial_n: C_n(X; R) \rightarrow C_{n-1}(X; R)$ to $A_n(X; R)$ obtaining $\partial_n: A_n(X; R) \rightarrow C_{n-1}(X; R)$. However, the restricted map need not have image in $A_{n-1}(X; R)$ (e.g. consider $X = I_2$). Hence, $A_\bullet(X; R)$ does not naturally form a chain complex. The next definition explains how to remedy this issue.

DEFINITION 2.17. Let ι_n denote the inclusion $A_n(X; R) \hookrightarrow C_n(X; R)$. Let $\Omega_0(X; R) = A_0(X; R)$. For all $n > 0$, define $\Omega_n(X; R)$ to be the pullback of the diagram:

$$\begin{array}{ccc} \Omega_n(X; R) & \longrightarrow & A_{n-1}(X; R) \\ \downarrow & \lrcorner & \downarrow \iota_{n-1} \\ A_n(X; R) & \xrightarrow{\partial_n \circ \iota_n} & C_{n-1}(X; R) \end{array}$$

Explicitly, the elements in $\Omega_n(X; R)$ consist of pairs $(a, b) \in A_n(X; R) \times A_{n-1}(X; R)$ such that $(\partial_n \iota_n)(a) = \iota_{n-1}(b)$ in $C_{n-1}(X; R)$.

DEFINITION 2.18. For each $n > 0$, let $\partial_n: \Omega_n(X; R) \rightarrow \Omega_{n-1}(X; R)$ be the map $\partial_n(a, b) := (b, 0)$.

LEMMA 2.19 ([15, §2.4]). With the definitions above, $(\Omega_\bullet(X; R), \partial_\bullet)$ is a chain complex.

Proof. We have $\partial_{n-1}(\partial_n(a, b)) = \partial_{n-1}(b, 0) = (0, 0)$ i.e. $\partial^2 = 0$. \square

DEFINITION 2.20. The path homology of a digraph X with R -coefficients, denoted $H_\bullet(X; R)$, is the homology of the chain complex $(\Omega_\bullet(X; R), \partial_\bullet)$.

DEFINITION 2.21. Given a ring R , a digraph map $f: X \rightarrow Y$ is an R -homology isomorphism if for all $n \geq 0$, the induced map $f_*: H_n(X; R) \rightarrow H_n(Y; R)$ is an isomorphism of R -modules.

If the ring R is clear from context or the statement is true for an arbitrary coefficient ring, we will speak of just *homology isomorphisms*. Likewise, we often write $\Omega_n(X)$, $H_n(X)$, etc. when the coefficient ring R is clear from context. Our main goal (Theorem 5.1) is to show that homology isomorphisms (for any ring R) are a ‘convenient’ class of weak equivalences in that they are a part of a cofibration category structure on DiGraph .

For the benefit of the readers unfamiliar with path homology, we now compute some examples of path homology by hand. Coefficients are in a general ring R .

EXAMPLE 2.22. Let I_2 be as in Definition 2.6. The regular allowed paths in I_2 as well as representatives for elements in $\Omega_\bullet(I_2)$ can be found in the table below. Although 012 is present in A_2 , its boundary is $12 - 02 + 01$ which does not land in A_1 since 02 is not an allowed path in A_1 . Thus $\Omega_2 = 0$.

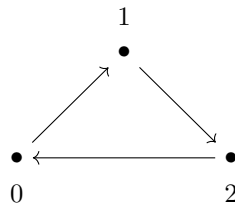
$$\begin{array}{ccccc}
 0 & & 1 & & 2 \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet
 \end{array}$$

| l | A_l | Ω_l | H_l |
|----------|-------------|-------------|-------|
| 0 | 0 1 2 | 0 1 2 | R |
| 1 | 01 12 | 01 12 | 0 |
| 2 | 012 | \emptyset | 0 |
| ≥ 3 | \emptyset | \emptyset | 0 |

The previous example is a simple case of the following general result:

LEMMA 2.23 ([17, Cor. 4.6]). *If the underlying graph of $X \in \text{DiGraph}$ is a tree, then $\Omega_l(X) = 0$ for all $l \geq 2$, and X has trivial path homology above degree 1.*

EXAMPLE 2.24 (Oriented triangle). Consider the cycle graph C_3 as in Definition 2.7.



| l | A_l | Ω_l | H_l |
|------------|--|-------------|-------|
| 0 | 0 1 2 | 0 1 2 | R |
| 1 | 01 12 20 | 01 12 20 | R |
| 2 | 012 120 201 | \emptyset | 0 |
| 3 | 0120 1201 2012 | \emptyset | 0 |
| $l \geq 4$ | $01 \dots l \quad 12 \dots l+1 \quad 20 \dots l+2$ | \emptyset | 0 |

The image of $\partial_1 : \Omega_1(C_3) \rightarrow \Omega_0(C_3)$ is generated by $0 - 1, 1 - 2$, and the kernel of ∂_1 is generated by $01 + 12 + 20$. To see that Ω_l is zero for $l \geq 2$, note that the boundary ∂_l of $(i, i + 1, i + 2, \dots, i + l)$ includes a nonzero summand

$$(i, i + 2, \dots, i + l) \in R\{(C_3)_V^l\},$$

which is not an allowed path and is not a summand of the boundary of any of the other generator $(j, j + 1, j + 2, \dots, j + l)$ for $A_l(C_3)$.

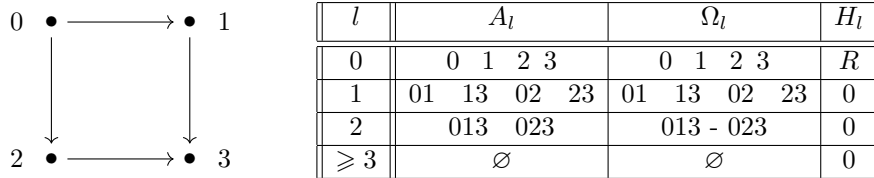
EXAMPLE 2.25 (Commuting triangle). Consider $C_{2,1}$ as in Definition 2.9.

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & \bullet & & \\
 & \nearrow & & \searrow & \\
 \bullet & & & & \bullet \\
 0 & \longrightarrow & & & 2
 \end{array}$$

| l | A_l | Ω_l | H_l |
|----------|-------------|-------------|-------|
| 0 | 0 1 2 | 0 1 2 | R |
| 1 | 01 12 02 | 01 12 02 | 0 |
| 2 | 012 | 012 | 0 |
| ≥ 3 | \emptyset | \emptyset | 0 |

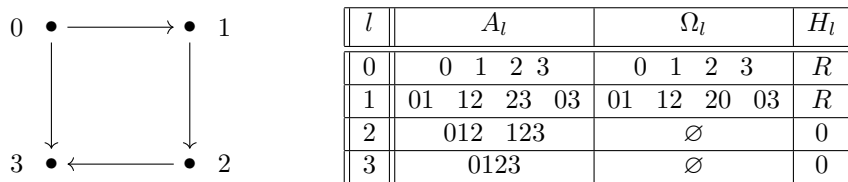
The image of $\partial_1 : \Omega_1(C_{2,1}) \rightarrow \Omega_0(C_{2,1})$ is generated by $1 - 0, 2 - 1$ so the kernel of ∂_2 on $\Omega_\bullet(C_{2,1})$ is one-dimensional. Its generator is given by $\partial_2(201) = 12 - 02 + 01$.

EXAMPLE 2.26 (Commuting Square). Consider $C_{2,2}$ as in Definition 2.9.



Despite looking like a the topological circle S^1 , H_1 of this graph is 0. Individually, the boundaries of the allowed paths 013 and 023 are not in A_1 , since A_1 does not contain 03. However, the boundary of the linear combination $013 - 023$ *does* land in A_1 . We thus have a single non-zero element in Ω_2 whose boundary generates the kernel of ∂_1 and $H_1 = 0$.

EXAMPLE 2.27. Consider the cycle $C_{3,1}$ as in Definition 2.9.

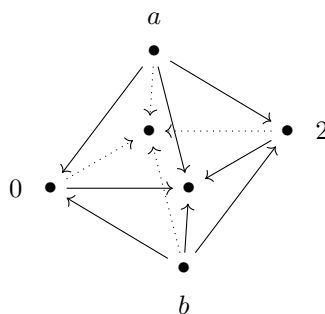


Unlike the previous example, for $l \geq 2$, there are no linear combinations of elements in A_l whose boundaries lie in A_{l-1} . Hence $\Omega_l = 0$ for $l \geq 2$. The kernel of ∂_1 is 1-dimensional and the image of ∂_2 is 0, so $H_1 = R$.

REMARK 2.28. Example 2.24 Example 2.25, Example 2.26, and Example 2.27 are explicit cases of [17, Prop. 4.3]. A cycle graph with at least three vertices and any orientation on edges has the homology type of S^1 unless it is the commuting triangle (Example 2.25) or the commuting square (Example 2.26).

The previous examples may give the impression that the homology of a digraph is always trivial above degree 1, but this is not the case.

EXAMPLE 2.29. Let $S\tilde{C}_4$ denote the digraph with vertices $a, 0, 1, 2, 3, b$ as depicted in the diagram below, where 1 and 3 are the two unlabelled vertices (it does not matter which):



| l | A_l | Ω_l | H_l |
|----------|---|---|-------|
| 0 | a 0 1 2 3 b | a 0 1 2 3 b | R |
| 1 | a0 a1 a2 a3 01 21 23 03 b0 b1 b2 b3 | a0 a1 a2 a3 01 21 23 03 b0 b1 b2 b3 | 0 |
| 2 | a01 a21 a23 a03 b01 b21 b23 b03 | a01 a21 a23 a03 b01 b21 b23 b03 | R |
| ≥ 3 | \emptyset | \emptyset | 0 |

Note that all length two allowed paths contribute to Ω_2 in this case. The image of ∂_1 is 5-dimensional so the kernel of ∂_1 is 7-dimensional. The image of ∂_2 is also 7-dimensional, so $H_1 = 0$. The kernel of ∂_2 is generated by $a01 - a21 + a23 - a03 - b01 + b21 - b23 + b03$ and, since $\Omega_3 = 0$, $H_2 = R$.

REMARK 2.30. The graph of Example 2.29 above can be thought of a suspension of \tilde{C}_4 and the result of the computation is closely related to [17, Prop. 5.10].

We have written a Python script to compute the dimensions of path homology over \mathbb{R} , which may be found at [8]. The script first generates the matrix representing the map:

$$A_n X \xrightarrow{\partial_n \circ \iota_n} C_{n-1} X \xrightarrow{\pi_{n-1}} C_{n-1} X / A_{n-1} X$$

It then computes a basis for the nullspace of this matrix, which is a basis of Ω_n by the following lemma:

LEMMA 2.31. $\Omega_n X$ is the kernel of the map $A_n X \rightarrow C_{n-1} X / A_{n-1} X$, i.e. we have an exact sequence:

$$0 \longrightarrow \Omega_n X \hookrightarrow A_n X \longrightarrow C_{n-1} X / A_{n-1} X$$

Proof. We first note that we have an exact sequence

$$0 \longrightarrow A_{n-1} X \xrightarrow{\iota_{n-1}} C_{n-1} X \xrightarrow{\pi_{n-1}} C_{n-1} X / A_{n-1} X \longrightarrow 0$$

This implies that the square on the right in the following diagram is a pullback (and a pushout, but we will not need this fact):

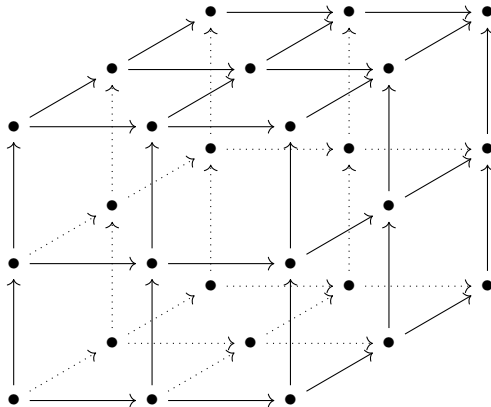
$$\begin{array}{ccccc} \Omega_n & \longrightarrow & A_{n-1} & \longrightarrow & 0 \\ \downarrow & \lrcorner & \downarrow \iota_{n-1} & \lrcorner & \downarrow \\ A_n & \xrightarrow{\partial_n \circ \iota_n} & C_{n-1} & \xrightarrow{\pi_{n-1}} & C_{n-1} / A_{n-1} \end{array}$$

We thus have a composite of two pullback squares, which is itself a pullback. The outer pullback square gives our desired exact sequence. \square

The matrix representing the differential $\partial_n : \Omega_n X \rightarrow \Omega_{n-1} X$ is given by restricting the map $A_n X \xrightarrow{\iota_n} C_n X \xrightarrow{\partial_n} C_{n-1} X$ to $\Omega_n X \rightarrow \Omega_{n-1} X$ and expressing its matrix in terms of the bases for $\Omega_n X$ and $\Omega_{n-1} X$. We compute the rank and nullity for the matrices of $\partial_1, \partial_2, \dots, \partial_K$, where K is some pre-defined cut-off. The nullity of ∂_0 is defined to be $\dim \Omega_0 X$. The dimensions of $H_n(X)$ for $0 \leq n < k$ are then given by

$$\dim H_n(X) = \text{nullity } \partial_n - \text{rank } \partial_{n+1}.$$

EXAMPLE 2.32. Consider the subgraph obtained by removing the central vertex of $I_2 \square I_2 \square I_2$:



| l | $\dim \Omega_l$ | H_l |
|-----|-----------------|-------|
| 0 | 26 | R |
| 1 | 48 | 0 |
| 2 | 24 | R |
| 3 | 0 | 0 |
| 4 | 0 | 0 |

Using our script, we calculate the dimensions of Ω_\bullet and H_\bullet , which are displayed in the table above. The 24 small square faces assemble to form a 2-dimensional “hole” that remains unfilled, since Ω_3 is 0. This is witnessed by the non-zero H_2 . By contrast $H_2(I_2 \square I_2 \square I_2) = 0$.

2.3. COFIBRATION CATEGORIES. We now introduce cofibration categories, a categorical framework for studying abstract homotopy theory. The origin of this notion, or more precisely its formal dual, goes back to Brown’s *categories of fibrant objects* [7], a notion that was introduced to study generalized sheaf cohomology. Many variations on the definition have appeared since, notably in Baues’ book [6] and Radulescu-Banu’s Ph.D. thesis [25].

Cofibration categories provide a way of speaking about homotopy theories with (finite) homotopy colimits. This statement was made precise by Szumilo [27], who showed that the homotopy theory of cofibration categories is equivalent to the homotopy theory of (finitely) cocomplete $(\infty, 1)$ -categories.

DEFINITION 2.33. A cofibration category consists of a category \mathcal{C} together with two classes of maps in \mathcal{C} : cofibrations, denoted \hookrightarrow , and weak equivalences, denoted $\xrightarrow{\sim}$, subject to the following conditions (where by an acyclic cofibration we mean a morphism that is both a cofibration and a weak equivalence):

- (C1) For any object $X \in \mathcal{C}$, the identity map id_X is an acyclic cofibration. Both cofibrations and weak equivalences are closed under composition.
- (C2) The class of weak equivalences satisfies the 2-out-of-6 property, i.e. given a triple of composable morphisms $f: X \rightarrow Y$, $g: Y \rightarrow Z$, and $h: Z \rightarrow W$, if gf and hg are weak equivalences, then so are f , g , h , and hgf .
- (C3) The category \mathcal{C} admits an initial object \emptyset and for any object $X \in \mathcal{C}$, the unique map $\emptyset \rightarrow X$ is a cofibration (i.e. all objects are cofibrant).
- (C4) The category \mathcal{C} admits pushouts along cofibrations. Moreover, the pushout of an (acyclic) cofibration is an (acyclic) cofibration.
- (C5) For any object $X \in \mathcal{C}$, the codiagonal map $X \sqcup X \rightarrow X$ can be factored as a cofibration followed by a weak equivalence.
- (C6) The category \mathcal{C} has small coproducts.
- (C7) The transfinite composite of (acyclic) cofibrations is again an (acyclic) cofibration.

The above definition most closely resembles the one given in [27] and is a slight strengthening of what might be found in [7, 6]. For instance, we require in (C2) that weak equivalences satisfy the 2-out-of-6 property instead of the (perhaps more common) 2-out-of-3 property. We recall that 2-out-of-6 implies 2-out-of-3.

LEMMA 2.34. *Weak equivalences in any cofibration category satisfy the 2-out-of-3 property, i.e. given a composable pair of maps $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, if any two of f , g , gf are weak equivalences, then so is the third.* \square

Furthermore, in the presence of axioms (C1), (C3), (C4), and (C5), 2-out-of-6 is equivalent to 2-out-of-3 with an additional requirement that the class of weak equivalences is saturated, i.e. that maps inverted when passing to the homotopy category are exactly the weak equivalences (this result is due to Cisinski, cf. [25, Thm. 7.2.7]).

The seven axioms presented above fall into two categories: axioms (C1)–(C5) correspond to finite homotopy colimits, whereas axioms (C6) and (C7) say that the homotopy theory additionally admits infinite homotopy colimits.

Before discussing examples of cofibration categories, we briefly record two consequences of the axioms. The first shows that the axiom (C5) can be strengthened to ask for factorizations of arbitrary maps rather than just the codiagonal morphism.

LEMMA 2.35 (Factorization Lemma, [7, p. 421]). *Every map f in \mathcal{C} can be factored as $f = wi$ where i is a cofibration and w is a weak equivalence.*

LEMMA 2.36 (Left Properness, [7, Lem. I.4.2]). *The pushout of a weak equivalence along a cofibration is again a weak equivalence.*

Readers familiar with homotopical algebra may recognize that a large class of examples of cofibration categories come from model categories. The latter, introduced by Quillen [24], were the first known way of abstractly capturing what a “homotopy theory” is. In brief, a model category is a complete and cocomplete category \mathcal{C} equipped with three classes of morphisms: cofibrations, fibrations, and weak equivalences subject to axioms similar to, although stronger than, the ones of a cofibration category. Given a model category \mathcal{C} , its full subcategory of cofibrant objects (i.e. objects X for which the map $\emptyset \rightarrow X$ is a cofibration) is a cofibration category. However, we shall not be concerned with model categories in this paper.

EXAMPLE 2.37. The category **Top** of topological spaces and some of its subcategories carry several interesting cofibration category structures:

- The *Hurewicz cofibration category* structure is defined on the category of all spaces. Its weak equivalences are the homotopy equivalences and its cofibrations are Hurewicz cofibrations, i.e. maps $i: A \hookrightarrow X$ such that, for any space S , any commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & S^{[0,1]} \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

where the right hand map is the evaluation at 0 admits a diagonal filler, i.e. there is a map $X \rightarrow S^{[0,1]}$ making both triangles commute. (This cofibration category structure arises from the Hurewicz model structure on **Top**, cf. [26, Thm. 3].)

- The *Serre cofibration category* structure is defined on the category of retracts of CW-complexes. Its weak equivalences are weak homotopy equivalences, i.e. maps inducing isomorphisms on homotopy groups, and its cofibrations are retracts of CW-inclusions. (This cofibration category structure arises from the Serre model structure on **Top**, cf. [20, Thm. 2.4.19].)
- The *Dold cofibration category* structure is defined on the category of all spaces. Its weak equivalences are the homotopy equivalences and its cofibrations are

Dold cofibrations, i.e. maps $A \rightarrow X$ satisfying the following weak homotopy extension condition: for any space S , every commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & S^{[0,1]} \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

admits a diagonal filler making the upper triangle commute strictly and the lower triangle commute up to a homotopy relative to A . (This cofibration category structure does *not* arise from a model structure, cf. [27, Thm. 1.32.(2)].)

EXAMPLE 2.38. For chain complexes Ch_R over a unital ring R :

- The *injective cofibration category* structure is defined on all chain complexes Ch_R . Its weak equivalences are quasi-isomorphisms, i.e. maps inducing isomorphisms on all homology groups, and its cofibrations are monomorphisms (cf. [20, Thm. 2.3.13]).
- The *projective cofibration category* structure is defined on chain complexes of projective R -modules $\text{Ch}_R^{\text{proj}}$. Its weak equivalences are once again the quasi-isomorphisms and its cofibrations are monomorphisms with degree-wise projective cokernels (cf. [20, Thm. 2.3.11]).

DEFINITION 2.39. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between cofibration categories is exact if it preserves cofibrations, acyclic cofibrations, the initial object, pushouts along cofibrations, coproducts, and transfinite composites of cofibrations.

REMARK 2.40. It follows by Ken Brown’s Lemma [20, Lem. 1.1.12] that exact functors preserve weak equivalences.

EXAMPLE 2.41. The inclusion $\text{Ch}_R^{\text{proj}} \hookrightarrow \text{Ch}_R$ is an exact functor from the projective cofibration category of chain complexes to the injective one.

EXAMPLE 2.42. The singular chain complex functor $C_\bullet: \text{Top} \rightarrow \text{Ch}_\mathbb{Z}$ taking a topological space to its singular complex is an exact functor. Here, Top is considered with the Serre cofibration category structure and $\text{Ch}_\mathbb{Z}$ is considered with the injective cofibration category structure. It can also be noted that the functor takes values in the category $\text{Ch}_\mathbb{Z}^{\text{proj}}$ and could also be considered as an exact functor into the projective cofibration category structure.

3. COFIBRATIONS OF DIRECTED GRAPHS

In this section, we define a suitable notion of cofibration of directed graphs (Definition 3.9) and prove some closure properties of these cofibrations (Proposition 3.12 through Proposition 3.18). Many of these properties correspond to cofibration category axioms. Stability of acyclic cofibrations under pushout requires a proof of the excision axiom, to which we dedicate Section 4. We defer the full proof that our cofibrations and weak equivalences comprise a cofibration category structure on DiGraph until Section 5. Unless otherwise noted, X and A (resp. Y and B , X' and A') will refer to a graph and an induced subgraph, respectively.

3.1. PROJECTING DECOMPOSITIONS. We begin this section by adapting the notion of projecting decomposition (cf. [23, Def. 4.6] and [19, Def. 26]) to the setting of directed graphs.

DEFINITION 3.1. For a directed graph X with an induced subgraph A , let X^A denote the induced subgraph on the set of vertices of X which admit a path to some vertex of A .

DEFINITION 3.2. For a vertex x of X^A , the height of x , denoted $h(x)$, is the minimal length of a path from x to a vertex of A .

DEFINITION 3.3. A projecting decomposition of X with respect to A is a function $\pi: X_V^A \rightarrow A_V$ such that, for any $x \in X_V$ and any $a \in A_V$ admitting a path from x , there is a path from x to a of minimal length which passes through πx .

EXAMPLE 3.4. Let X be the cycle $C_{3,1}$ from Definition 2.9. If A is the edge $0 - 1$ (Figure 8a), the only vertices in X admitting a path to A are those in A itself, so $X_V^A = A_V$ and the identity function is a projecting decomposition.

If A is the edge $1 - 2$ (Figure 8b), X^A is the induced subgraph on the vertices $0, 1$ and 2 , with a projecting decomposition sending 0 and 1 to 1 , and 2 to itself.

If A is the edge $2 - 3$ (Figure 8c), X^A is the whole of X . However, X admits no projecting decomposition π to A : if $\pi(0) = 2$, the unique path of minimal length 1 from 0 to 3 does not pass through 2 ; and if $\pi(0) = 3$, the unique path of minimal length 2 from 0 to 2 does not pass through 3 .

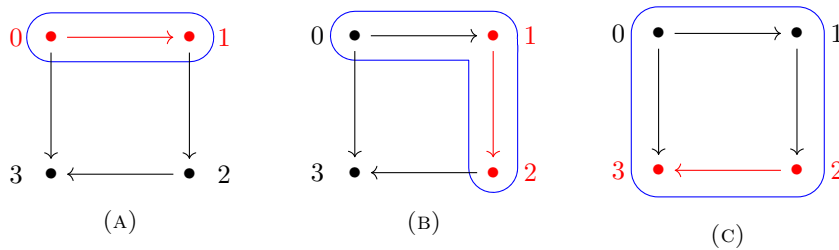


FIGURE 8. The graph $X = C_{3,1}$ with induced subgraphs A (in red) and corresponding X^A (circled in blue). Only the first two examples admit projecting decompositions.

We now explore some basic consequences of the definition of a projecting decomposition.

LEMMA 3.5. If X admits a projecting decomposition with respect to A , then for any $x \in X_V^A$, πx is the unique vertex of A which is closest to x , i.e. the unique vertex of A admitting a path of length $h(x)$.

In particular, if $x \in A_V$, then $\pi x = x$, while if $h(x) = 1$ then πx is the unique vertex of A admitting an edge from x .

Proof. Suppose that $a \in A_V$ admits a path of length $h(x)$ from x to a . By the definition of a projecting decomposition, there exists a path of length $h(x)$ from x to a which passes through πx . As $\pi x \in A_V$, the minimality of $h(x)$ implies that πx must be the final vertex of this path and in fact $\pi x = a$. \square

COROLLARY 3.6. If X admits a projecting decomposition with respect to A , then it is unique. \square

REMARK 3.7. We will sometimes view a projecting decomposition π as a function on $X_V^A \setminus A_V$ rather than X_V^A , when this suits our computational purposes. This abuse of terminology is justified by Lemma 3.5, which implies that a function on $X_V^A \setminus A_V$ satisfying the criteria of Definition 3.3 extends uniquely to a projecting decomposition of X with respect to A , by setting $\pi a = a$ for all $a \in A_V$.

LEMMA 3.8. Suppose X admits a projecting decomposition with respect to A . Let $x \rightarrow y$ denote an edge of X , with both x and y admitting paths to A , and $h(x) \geq h(y)$. Then one of the following conditions holds:

- (1) $h(x) = h(y)$ and there exists an edge $\pi x \rightarrow \pi y$;
- (2) $h(x) = h(y) + 1$ and $\pi x = \pi y$.

Proof. If $x = y$ then condition 1 is trivially satisfied, so assume otherwise. We first note that by Lemma 3.5, there exists a path of length $h(y)$ from y to πy . Concatenating this path with the edge $x \rightarrow y$, we obtain a path from x to πy of length $h(y) + 1$. It thus follows that $h(x) \leq h(y) + 1$.

Moreover, by the definition of a projecting decomposition, there is a path of minimal length from x to πy which passes through πx ; the length of this path must be less than or equal to $h(y) + 1$. The assumption that $h(x) \geq h(y)$ then implies that its length is either $h(y)$ or $h(y) + 1$.

First, suppose the length of this path is $h(y)$. Then $h(x) \leq h(y)$; together with our assumption on $h(x)$ this implies $h(x) = h(y)$. By Lemma 3.5 we see that $\pi x = \pi y$, so that condition 1 is satisfied in this case.

Next suppose the length of the chosen path from x to πy is $h(y) + 1$. If $\pi x = \pi y$, then this implies that $h(x) = h(y) + 1$, so that condition 2 is satisfied. Otherwise, consider the initial segment of this path from x to πx . As the path's length is minimal, this initial segment must have length $h(x)$. By our assumption that $\pi x \neq \pi y$, this initial segment is not the entire path; thus we see that $h(x) \leq h(y)$, once again implying $h(x) = h(y)$. As the length of the entire path is $h(y) + 1$, the path consists of this initial segment concatenated with an edge from πx to πy . Thus condition 1 is satisfied in this case. □

3.2. DEFINITION OF COFIBRATIONS. We now define the cofibrations which will be our objects of study.

DEFINITION 3.9. A cofibration of directed graphs is an induced subgraph inclusion $A \hookrightarrow X$ satisfying the following two conditions:

- there are no edges out of A , i.e. no edges from the vertices of A to those not contained in A ;
- X admits a projecting decomposition with respect to A .

EXAMPLE 3.10. The inclusion of the edge $2 - 3$ in the commuting square $C_{2,2}$ is a cofibration: there are no edges out of $2 - 3$, and the map $0 \mapsto 2, 1 \mapsto 3$ is a projecting decomposition.

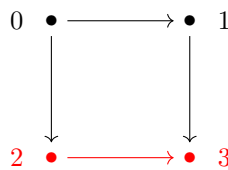


FIGURE 9. The inclusion of the edge $2-3$ (in red) into the commuting square $X = C_{2,2}$ is a cofibration.

REMARK 3.11. One might ask whether a simpler cofibration category for path homology can be obtained using only one of the two conditions of Definition 3.9 to define cofibrations. In fact, both conditions are necessary.

To see this, consider the inclusion of the edge $0 - 1$ or the edge $2 - 3$ into the $(3, 1)$ -cycle $C_{3,1}$. The edge $0 - 1$ (Fig. 8a) admits a projecting decomposition, as we saw in Example 3.4, but also admits edges out of itself. Meanwhile, the edge $2 - 3$

(Fig. 8c) does not admit edges out of itself, but also does not admit a projecting decomposition. These examples would be cofibrations if we omit either of the conditions in the definition.

In each case, consider the pushout of the inclusion of the edge in question along the homology isomorphism $I_1 \rightarrow I_0$ (see Lemma 2.23). The pushouts are both isomorphic in DiGraph to $C_{2,1}$ of Definition 2.9. The map from $C_{3,1}$ to the pushout cannot be a homology isomorphism (Examples 2.25 and 2.27). Thus the proposed cofibration category structure does not exist, as it fails left properness (Lemma 2.36).

The remainder of this section will be concerned with proving certain useful properties of cofibrations. In particular, many of these results will be used in Section 5 to establish a cofibration category structure on DiGraph , with cofibrations as defined above and path homology isomorphisms as the weak equivalences.

PROPOSITION 3.12. *The class of cofibrations contains all identities and is closed under composition.*

Proof. To see that all identities are cofibrations, we note that in the case $A = X$, it is trivially true that there are no edges out of A and a projecting decomposition given by the identity on A_V .

Now let $A \rightarrow X$ and $X \rightarrow Y$ be a composable pair of cofibrations; we must show that the composite inclusion $A \rightarrow Y$ is a cofibration.

First, let y denote a vertex of $Y \setminus A$ and let a denote a vertex of A ; we will show that there is no edge from a to y . If $y \in X_V$ then this follows from the fact that $A \rightarrow X$ is a cofibration, since there are no edges out of A in X ; similarly, if $y \in Y_V \setminus X_V$ then this follows from the fact that $X \rightarrow Y$ is a cofibration.

Now we construct a projecting decomposition of Y with respect to A . Note that we already have projecting decompositions of Y with respect to X and of X with respect to A ; denote these by π_X and π_A , respectively.

For a vertex $y \in Y_V^A$, let a denote a vertex of A admitting a path from y . Then since $a \in X_V$, there is a path from y to a of minimal length passing through $\pi_X y$. Observe that the terminal segment of this path beginning at $\pi_X y$ defines a path in X of minimal length from $\pi_X y$ to a ; thus we may replace this segment with one of equal length passing through $\pi_A \pi_X y$. We can therefore define πy to be $\pi_A \pi_X y$. Note that this definition does not depend on the choice of $a \in A_V$ admitting a path from y . It follows that π is a projecting decomposition of Y with respect to A , as for any vertex a of A admitting a path from y , we have defined a path of minimal length from y to a which passes through πy . \square

PROPOSITION 3.13. *For every directed graph X , the unique map $\emptyset \rightarrow X$ is a cofibration.*

Proof. It is trivially true that there are no edges out of the empty subgraph of X . To obtain a projecting decomposition, we note that both \emptyset_V and A_V^\emptyset are empty, so a suitable function π is given by the identity on the empty set. \square

3.3. CLOSURE PROPERTIES OF COFIBRATIONS. In this subsection, we prove that cofibrations are closed under several natural operations: box products, pushouts, and retracts. Several negative results are provided as well, including the failure of monoidality and non-existence of a small generating set of cofibrations.

PROPOSITION 3.14. *A box product of cofibrations is a cofibration.*

Proof. Let $A \rightarrow X$ and $B \rightarrow Y$ be a pair of cofibrations. Consider the box product $A \square B \rightarrow X \square Y$. By the definition of the box product, we may observe that the

vertices of $X \square Y$ consist of all pairs (x, y) with $x \in X_V, y \in Y_V$, and that $A \square B$ is the induced subgraph on the set of such pairs for which $x \in A_V, y \in B_V$. Given two vertices $(x, y), (x', y')$, there is an edge $(x, y) \rightarrow (x', y')$ if and only if there are edges $x \rightarrow x'$ in H and $y \rightarrow y'$ in Q , with at least one of these edges being degenerate (i.e. either $x = x'$ or $y = y'$).

It follows that a path from (x, y) to (x', y') is equivalent to an interleaving of a path from x to x' in X with a path from y to y' in Y . More precisely, such a path consists of a sequence of edges of $X \square Y$, each necessarily of the form (e_i, id) for e_i an edge of X or (id, e'_i) for e'_i an edge of Y , where the sequence of edges (e_i) forms a path from x to x' in X and the sequence of edges (e'_i) forms a path from y to y' in Y . The length of such a path is the sum of the lengths of its component paths. This characterization of edges in $X \square Y$ shows that there are no edges out of $A \square B$.

Now we define a projecting decomposition of $X \square Y$ with respect to $A \square B$. For (x, y) admitting a path to a vertex of $A \square B$, we define $\pi(x, y) = (\pi x, \pi y)$. To see that this is a projecting decomposition, consider a path in $X \square Y$ from (x, y) to (a, b) where $a \in A_V, b \in B_V$. The characterization of paths above shows that this path is obtained by interleaving a path from x to a in X with a path from y to b in Y . From these we obtain paths of minimal length from x to a through πx , and from y to b through πy . Choosing a suitable interleaving of these paths, we obtain a path of minimal length from (x, y) to (a, b) passing through $(\pi x, \pi y)$. Concretely, we may construct the desired path as follows:

- proceed from (x, y) to $(\pi x, y)$, moving at each step along the chosen path from x to a in the first component while keeping the second component fixed;
- proceed from $(\pi x, y)$ to $(\pi x, \pi y)$, moving at each step along the chosen path from y to b in the second component while keeping the first component fixed;
- proceed similarly from $(\pi x, \pi y)$ to $(a, \pi y)$;
- proceed similarly from $(a, \pi y)$ to (a, b)

Thus we see that π is indeed a projecting decomposition. □

PROPOSITION 3.15. *Cofibrations are stable under pushout.*

Proof. Consider a pushout diagram of directed graphs as depicted below, with $A \twoheadrightarrow X$ a cofibration.

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & A' \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

Applying Lemma 2.10 and Corollary 2.11, we see that there are no edges out of A' in X' (as there are no edges out of A in X), and that the complement of A' in X' is isomorphic to the complement of A in X .

We now define a projecting decomposition π' of X' with respect to A' . Note that, by Corollary 2.11, a vertex of $X' \setminus A'$ admits a path to A' if and only if the corresponding vertex of $X \setminus A$ admits a path to A . Thus we define π' as follows: for $a \in A'_V$ we set $\pi' a = a$, while for $x \in (X')^{A'}_V \setminus A'_V$ we set $\pi' x = f(\pi x)$.

It remains to show that π' is in fact a projecting decomposition. Take $x \in (X')^{A'}_V \setminus A'_V$ and let p be a path in X' from x to $a \in A'$. Without loss of generality assume that p is of minimal length among such paths. Since there are no edges out of A' , the path p may be depicted as in the following diagram, where $y_i \in (X')^{A'}_V \setminus A'_V$ for all i and $b_j \in A'_V$ for all j :

$$\begin{array}{cccccccccccc}
 x & & y_1 & & & & y_m & & b_1 & & & & b_n & & a \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet
 \end{array}$$

By Lemma 2.10, the existence of the edge $y_m \rightarrow b_1$ implies the existence of an edge $y_m \rightarrow \bar{b}_1$ in X , with $\bar{b}_1 \in A_V$, such that $f(\bar{b}_1) = b_1$. By the fact that π is a projecting decomposition, it follows that there is a path of minimal length from x to \bar{b}_1 in X which passes through πx . In fact, the assumption that p is of minimal length, together with Lemma 2.10, implies that the minimal length of a path from x to \bar{b}_1 in X is precisely $m + 1$, as any shorter path in X would induce a shorter path in X' . Thus we may write this path from x to \bar{b}_1 as:

$$\begin{array}{cccccccccccc}
 x & & z_1 & & & & z_k & & \pi x & & c_1 & & & & c_l & & \bar{b}_1 \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet
 \end{array}$$

where $z_i \in X_V \setminus A_V$ for all i , $c_j \in A_V$ for all j , and $k + l + 1 = m$. Thus we obtain the following path in X' :

$$\begin{array}{cccccccccccc}
 x & & z_1 & & & & z_k & & f(\pi x) & & f(c_1) & & & & f(c_l) & & f(\bar{b}_1) = b_1 & & b_n & & a \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet
 \end{array}$$

The length of this path is at most $k + l + n + 2 = m + n + 1$, the length of the original path p . As p was assumed to be of minimal length, we have thus constructed a path of minimal length from x to a passing through $\pi'x = f(\pi x)$. \square

As an aside, we mention that our cofibrations are not stable under the operation of taking *pushout-(box) products*, meaning that the cofibration category structure of Theorem 5.1 is not monoidal with respect to the box product. (Since this is only a comment, we do not recall the definition of a monoidal cofibration category in full detail.) This sets our cofibration category of directed graphs apart from many familiar cofibration category structures, e.g. the Serre cofibration category structure on topological spaces of Example 2.37, which is monoidal with respect to the cartesian product.

PROPOSITION 3.16. *Let $A \twoheadrightarrow X$ and $B \twoheadrightarrow Y$ be cofibrations such that $X \setminus A$ contains a vertex admitting a path to A , and likewise $Y \setminus B$ contains a vertex admitting a path to B . Then the pushout box product $X \square B \cup_{A \square B} A \square Y \twoheadrightarrow X \square Y$ is not a cofibration.*

Proof. By assumption, we have a vertex $x \in X_V$, not contained in A , admitting an edge to a vertex $a \in A_V$, and likewise we have a vertex $y \in Y_V$, not contained in B , admitting an edge to a vertex $b \in B_V$. The vertex (x, y) of $X \square Y$ is not contained in $X \square B \cup_{A \square B} A \square Y$. However, this vertex admits edges to the two distinct vertices $(a, y), (x, b)$ of $X \square B \cup_{A \square B} A \square Y$. By Lemma 3.5, it follows that $A \square Y$ does not admit a projecting decomposition with respect to $X \square B \cup_{A \square B} A \square Y$. \square

We next prove that our cofibrations, like the cofibrations of a model category, are closed under retracts and transfinite composition.

PROPOSITION 3.17. *Let $A \twoheadrightarrow X$ be a cofibration and let $B \rightarrow Y$ be a retract of $A \twoheadrightarrow X$, as depicted below:*

$$\begin{array}{ccccc}
 B & \xrightarrow{i} & A & \xrightarrow{f} & B \\
 \downarrow & & \downarrow & & \downarrow \\
 Y & \xrightarrow{j} & X & \xrightarrow{g} & Y
 \end{array}$$

where $fi = \text{id}_B$ and $gj = \text{id}_Y$. Then $B \rightarrow Y$ is a cofibration.

Proof. The fact that $A \rightarrow X$ is a cofibration implies that all maps in the left-hand square are inclusions on vertices, so we will consider vertices in B as vertices in A, Y , or X without relabeling.

We first show that $B \rightarrow Y$ is an induced subgraph inclusion. Consider vertices $b, b' \in B_V$ such that there is an edge $b \rightarrow b'$ in Y . Then there is an edge $b \rightarrow b'$ in X , hence also in A as A is an induced subgraph of X . Taking the image of this edge under the retraction $f: A \rightarrow B$ gives an edge $b \rightarrow b'$ in B .

A similar argument shows that there are no edges out of B in Y . Explicitly, consider a pair of vertices $b \in B_V, y \in Y_V$ with an edge $b \rightarrow y$. Then $b \in A$, implying that $y \in A$ as well since there are no edges out of A in X . It follows that $g(y) = y$ is contained in B , as $f: A \rightarrow B$ is the restriction of g to A .

Finally, we will show that Y admits a projecting decomposition with respect to B . Let π denote the projecting decomposition of X with respect to A ; we will show that $\pi|_{Y^\beta}$ is a projecting decomposition of Y with respect to B .

Let y be a vertex of Y admitting a path to some vertex b of B . We first note that by Lemma 3.5, πy is the unique vertex of A admitting a path from y of length $h(y)$ in X . The image of this path under g defines a path from y to $f\pi y$ in $Y \subseteq X$ of length less than or equal to $h(y)$; by the minimality of $h(y)$ and uniqueness of πy , it follows that $f\pi y = \pi y$. This in turn implies that $\pi y \in B$.

Now observe that by the definition of a projecting decomposition, there is a path in X from y to b , of minimal length among such paths, which passes through πy . Denote the length of this path by n ; then the image of this path under g defines a path from y to b in Y , passing through $f\pi y$, of length less than or equal to n (hence equal to n by minimality). Moreover, as n is the minimum length of a path in X from y to b , it is likewise the minimum length of such paths in $Y \subseteq X$. Thus $\pi|_Y$ is indeed a projecting decomposition of Y with respect to B . \square

PROPOSITION 3.18. *The class of cofibrations is closed under transfinite composition.*

Proof. Let α be an arbitrary limit ordinal. Consider a diagram $X: \alpha \rightarrow \text{DiGraph}$ as depicted below:

$$X_0 \rightarrow X_1 \rightarrow \dots$$

in which each map $X_\beta \rightarrow X_{\beta+1}$ is a cofibration. For every limit ordinal $\beta < \alpha$, let X_β denote the union of all X_γ for $\gamma < \beta$, i.e. the colimit of the restricted diagram $X|_\beta$. Let X_α denote the colimit of this diagram, so that the map $X_0 \rightarrow X_\alpha$ is the transfinite composite of the maps $X_\beta \rightarrow X_{\beta+1}$. We will show, by induction on $\beta \leq \alpha$, that for each $\gamma < \beta$ the map $X_\gamma \rightarrow X_\beta$ is a cofibration; in particular, this implies that $X_0 \rightarrow X_\alpha$ is a cofibration.

In the base case $\beta = 0$, there is no $\gamma < \beta$, so the statement is vacuously true.

Let $\beta > 0$ and suppose the statement holds for all $\beta' < \beta$. In the case of a successor ordinal $\beta = \beta' + 1$, the transfinite composite factors as $X_\gamma \rightarrow X_{\beta'} \rightarrow X_{\beta'+1}$. If $\gamma = \beta'$ then the map $X_\gamma \rightarrow X_{\beta'}$ is the identity, while if $\gamma < \beta'$ then $X_\gamma \rightarrow X_{\beta'}$ is a cofibration by the induction hypothesis. The map $X_{\beta'} \rightarrow X_{\beta'+1}$ is a cofibration by assumption and so the composite is a cofibration by Proposition 3.12.

Now we consider the case of a limit ordinal β , so that X_β is the union of all the directed graphs $X_{\beta'}$ for $\beta' < \beta$. We first show that $X_\gamma \rightarrow X_\beta$ is an induced subgraph inclusion. Let x, x' be a pair of vertices of X_γ such that X_β contains an edge $x \rightarrow x'$. Then there is an edge $x \rightarrow x'$ in $X_{\beta'}$ for some $\gamma \leq \beta' < \beta$. Since $X_\gamma \rightarrow X_{\beta'}$ is a cofibration by the induction hypothesis, and hence an induced subgraph inclusion, there is an edge $x \rightarrow x'$ in X_γ as well.

Next let x be a vertex of X_γ and x' a vertex of X_β not contained in X_γ ; we will show there is no edge $x' \rightarrow x$ in X_β . Note that x' is a vertex of $X_{\beta'}$ for some $\beta' < \beta$, and $X_\gamma \twoheadrightarrow X_{\beta'}$ is a cofibration by the induction hypothesis. Thus there is no edge $x' \rightarrow x$ in $X_{\beta'}$; since we have just proven that $X_{\beta'} \rightarrow X_\beta$ is an induced subgraph inclusion, the same is true in X_β .

Finally, we show that X_β admits a projecting decomposition π with respect to X_γ . By Remark 3.7, it suffices to define π satisfying the criteria of Definition 3.3 on $x \in (X_\beta)_V^{X_\gamma} \setminus (X_\gamma)_V$. Let β' be minimal such that $x \in X_{\beta'}$. We must have that $\gamma < \beta' < \beta$. By the induction hypothesis, the inclusion $X_\gamma \twoheadrightarrow X_{\beta'}$ is a cofibration and thus admits a projecting decomposition $\pi_{\beta'}$. We define $\pi x = \pi_{\beta'} x$.

Note that any path in X_β from $x \in (X_{\beta'})_V$ to $y \in (X_\gamma)_V$ must be contained in $X_{\beta'}$, since there are no edges from $X_{\beta'}$ to X_β . Therefore a minimal length path from x to y in $X_{\beta'}$ which passes through $\pi_{\beta'} x$ is also of minimal length in X_β and passes through $\pi x = \pi_{\beta'} x$. \square

In view of the results above, given a set of cofibrations, we may consider the class of cofibrations which it generates under pushout, retract, and transfinite composition. We might naturally hope to find some set of cofibrations which generates the entire class of cofibrations in DiGraph via the procedure outlined above; however, Proposition 3.20 below shows that this is not possible. Since we are not working with a model category structure, we must first define our concept of generation under these operations precisely.

DEFINITION 3.19. *Let C denote a set of cofibrations in DiGraph . For each ordinal α , we define a class of cofibrations $\text{Gen}_\alpha(C)$ by transfinite induction on α as follows.*

- $\text{Gen}_0(C) = C$;
- for a successor ordinal $\alpha = \beta + 1$, $\text{Gen}_\alpha(C)$ is the class of all retracts, pushouts, and transfinite composites of maps in $\text{Gen}_\beta(C)$. In particular, this implies that $\text{Gen}_\beta(C)$ is contained in $\text{Gen}_\alpha(C)$;
- for a limit ordinal α , $\text{Gen}_\alpha(C)$ is the class of all maps contained in some $\text{Gen}_\beta(C)$ for $\beta < \alpha$.

Note that in general, $\text{Gen}_\alpha(C)$ will be a proper class for $\alpha > 0$.

We then define $\text{Gen}(C)$, the class of cofibrations generated by C , as the union of all the classes $\text{Gen}_\alpha(C)$, i.e. the class of all maps contained in $\text{Gen}_\alpha(C)$ for some α .

PROPOSITION 3.20. *There is no set of cofibrations C such that $\text{Gen}(C)$ is the class of all cofibrations.*

Proof. Let C denote an arbitrary set of cofibrations. Let S denote a set whose cardinality is greater than that of the vertex set of the codomain of any map in C and let K_S denote the complete directed graph having S as its set of vertices. Given $A \twoheadrightarrow X$, let $X - A$ denote the induced subgraph on $X_V \setminus A_V$.

By transfinite induction on ordinals α , we will show that $\text{Gen}_\alpha(C)$ does not contain any cofibration $A \twoheadrightarrow X$ such that $X - A$ contains K_S as a subgraph. (For instance, we may consider $A = \emptyset, X = K_S$.)

In the base case $\alpha = 0$, this is immediate from the definition of S .

Now suppose that the statement holds for all $\beta < \alpha$. If α is a limit ordinal, then it is immediate from the induction hypothesis that the statement holds for α as well. Now consider the case of a successor ordinal $\alpha = \beta + 1$. Given a cofibration $A \twoheadrightarrow X$ with an induced subgraph inclusion $K_S \subseteq X - A$, to show that $A \twoheadrightarrow X$ is not contained in $\text{Gen}_\alpha(C)$, we must show that it is not a retract, pushout, or transfinite composite of maps in $\text{Gen}_\beta(C)$.

For the case of retracts, suppose that $A \hookrightarrow X$ is a retract of some cofibration $B \hookrightarrow Y$, as depicted below.

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{f} & A \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{j} & Y & \xrightarrow{g} & X \end{array}$$

Then we have a subgraph inclusion $K_S \subseteq X \subseteq Y$. If some vertex s of K_S is contained in B , then $f(s) = s$ is contained in A . This contradicts our assumption that K_S is contained entirely in $X \setminus A$. Thus we see that K_S is contained in the complement $Y \setminus B$. By the induction hypothesis, it follows that $B \hookrightarrow Y$ is not contained in $\text{Gen}_\beta(C)$.

Next suppose that $A \hookrightarrow X$ is a pushout of some cofibration $B \hookrightarrow Y$, as depicted below.

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \quad \square$$

By assumption, no vertex of K_S is contained in A , so K_S is contained entirely in the image of $Y \rightarrow X$. Thus, by Lemma 2.10, Y contains K_S as a subgraph. Moreover, no vertex of $K_S \subseteq Y$ can be contained in B , as this would imply that the corresponding vertex of X was contained in A by the commutativity of the diagram. It follows that K_S is a subgraph of $Y \setminus B$, implying that $B \hookrightarrow Y$ is not contained in $\text{Gen}_\beta(C)$ by the induction hypothesis.

Now suppose that $A \hookrightarrow X$ is a transfinite composite of some family of cofibrations $X_\gamma \rightarrow X_{\gamma+1}$, indexed by $\gamma < \delta$ for some ordinal δ , with $X_0 = A$. Then for each such γ we have $X_\gamma \subseteq X$, and X is the union of all the X_γ ; in particular, each vertex of X is contained in some subgraph X_γ . Moreover, for $\rho < \gamma$ there are no edges from the vertices of $X_\gamma \setminus X_\rho$ to those of X_ρ , as the map $X_\rho \rightarrow X_\gamma$ is a cofibration by Proposition 3.18.

Let s denote an arbitrary vertex of $K_S \subseteq X$, and let γ be minimal such that s is a vertex of X_γ . If γ were a limit ordinal, then X_γ would be the union of all X_ρ for $\rho < \gamma$, contradicting minimality. Thus $\gamma = \gamma' + 1$ for some γ' such that s is not contained in $X_{\gamma'}$. As every other vertex of K_S admits edges to and from s , all vertices of K_S must also appear in X_γ , and not in $X_{\gamma'}$. As $X_\gamma \hookrightarrow X$ is an induced subgraph inclusion, it follows that K_S is a subgraph of $X_\gamma \setminus X_{\gamma'}$. By the induction hypothesis, the cofibration $X_{\gamma'} \hookrightarrow X_\gamma$ is not contained in $\text{Gen}_\beta(C)$.

Thus we see that $A \hookrightarrow X$ is not a retract, pushout or transfinite composite of any map in $\text{Gen}_\beta(C)$, so it is not contained in $\text{Gen}_\alpha(C)$.

By induction, no cofibration $A \hookrightarrow X$ such that K_S is a subgraph of $X \setminus A$ is contained in $\text{Gen}(C)$. Thus $\text{Gen}(C)$ is not the class of all cofibrations. \square

4. EXCISION

In the present section, we formulate and prove the excision axiom for path homology in terms of the cofibrations of Definition 3.9. This plays a crucial role in Section 5 when we establish a cofibration category structure on DiGraph (Theorem 5.1).

We now turn our attention to the *relative homology modules* $H_n(X, A)$ associated to a cofibration $A \hookrightarrow X$. Throughout this section, we fix a commutative coefficient ring R .

As our analysis of these groups will involve several constructions which are functorial with respect to commuting squares of cofibrations, we define DiGraph_2 to be

the category with cofibrations in DiGraph as its objects and commuting squares as its morphisms. Let $\text{DiGraph}_2^{\text{PO}}$ denote the subcategory of DiGraph_2 which contains all objects, but with only pushout squares as morphisms. Throughout this section, a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

i.e. a morphism in DiGraph_2 or $\text{DiGraph}_2^{\text{PO}}$, will be denoted by $f: (X, A) \rightarrow (Y, B)$.

4.1. STATEMENT OF EXCISION AXIOM. The goal of this short subsection is to give a statement of the ‘excision axiom’ (Theorem 4.5), namely that relative homology takes homotopy pushouts to isomorphisms.

DEFINITION 4.1. *Given a subgraph inclusion $A \subseteq X$, the n^{th} relative homology module $H_n(X, A)$ is the n^{th} homology module of the factor complex $\Omega(X)/\Omega(A)$.*

Given a cofibration $A \hookrightarrow X$, we let $X - A$ denote the complement of A in X , i.e. the induced subgraph on the vertices of X not contained in A . We define $(X - A)^A$ to be the induced subgraph of $X - A$ on the vertices which admit paths to A . We let $(X - A)_1$ denote the induced subgraph of X on the vertices of height 1, i.e. those vertices not in A which admit edges into A .

DEFINITION 4.2. *Let $Q: \text{DiGraph}_2 \rightarrow \text{Ch}_R$ denote the functor which sends a cofibration $A \hookrightarrow X$ to the factor complex $\Omega(X)/\Omega(A)$, and a morphism of cofibrations to the induced map between their factor complexes.*

We observe that the composite of Q with the homology functor $H_*: \text{Ch}_R \rightarrow \text{Mod}_R^{\mathbb{N}}$ sends each cofibration to its family of relative homology modules, and each commuting diagram to its family of induced maps on relative homology modules.

PROPOSITION 4.3 ([14, Thm. 3.11]). *For any subgraph inclusion $A \hookrightarrow X$, there is a relative homology long exact sequence:*

$$\cdots \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A) \rightarrow 0.$$

COROLLARY 4.4. *A subgraph inclusion $A \hookrightarrow X$ is a homology isomorphism if and only if all relative homology modules $H_n(X, A)$ are zero.* \square

Our main goal for this section is to prove the following.

THEOREM 4.5. *Given a pushout diagram $f: (X, A) \rightarrow (X', A')$ in $\text{DiGraph}_2^{\text{PO}}$, the induced map of relative homology modules $H_n(X, A) \rightarrow H_n(X', A')$ is an isomorphism for all $n \geq 0$.*

In Section 5, we will use this result to prove that cofibrations and homology isomorphisms form a cofibration category structure on DiGraph .

4.2. THE COMPLEMENT CHAIN COMPLEX. Note that Theorem 4.5 is equivalent to the statement that the composite functor $H_*Q: \text{DiGraph}_2 \rightarrow \text{Mod}_R^{\mathbb{N}}$ sends all pushout squares to isomorphisms of $\text{Mod}_R^{\mathbb{N}}$. Our strategy for proving the latter is as follows: we first define the ‘complement chain complex’ functor $\widehat{\Omega}: \text{DiGraph}_2 \rightarrow \text{Ch}_R$, and establish a natural isomorphism $Q \cong \widehat{\Omega}$. In the next subsection, we define another functor $M: \text{DiGraph}_2^{\text{PO}} \rightarrow \text{Ch}_R$, show that M sends all morphisms of $\text{DiGraph}_2^{\text{PO}}$ (i.e. all pushout squares) to isomorphisms of Ch_R , and establish a natural isomorphism $\widehat{\Omega}|_{\text{DiGraph}_2^{\text{PO}}} \cong M$.

We begin by defining the intermediate functor $\widehat{\Omega}$.

DEFINITION 4.6. Let $A \hookrightarrow X$ be an induced subgraph inclusion. For $n \geq 0$, the R -module $\widehat{A}_n(X, A)$ is the submodule of $A_n(X)$ generated by the paths which intersect the complement $X - A$. The R -module $\widehat{\Omega}_n(X, A)$ is defined as the intersection $\Omega_n(X) \cap \widehat{A}_n(X, A)$. The R -modules $\widehat{\Omega}_n(X)$ assemble to a chain complex $\widehat{\Omega}(X, A)$ as follows: for $\omega \in \widehat{\Omega}_n(X, A)$, the boundary $\partial\omega$ is computed by first computing the boundary of ω as an element of $\Omega_n(X)$, then setting any terms corresponding to paths not intersecting $X - A$ to 0.

LEMMA 4.7. Let $A \hookrightarrow X$ be a cofibration of directed graphs. Then for each $n \geq 0$, the inclusion $\Omega_n(A) \hookrightarrow \Omega_n(X)$ is isomorphic to the direct summand inclusion $\Omega_n(A) \hookrightarrow \Omega_n(A) \oplus \widehat{\Omega}_n(X, A)$.

Proof. This statement is essentially a strengthening of (the dual of) [14, Lem. 3.10]. The proof supplied in that reference shows that any linear combination $\omega \in \Omega_n(X)$ decomposes uniquely as $p(\omega) + q(\omega)$, where $p(\omega) \in \Omega_n(A)$ and $q(\omega) \in \widehat{\Omega}_n(X, A)$, and the maps p and q thus obtained are R -module homomorphisms. Thus we obtain an isomorphism $\Omega_n(X) \rightarrow \Omega_n(A) \oplus \widehat{\Omega}_n(X, A)$ sending each ω to $(p(\omega), q(\omega))$. The pre-image of the summand $\Omega_n(A)$ under this isomorphism is precisely $\Omega_n(A)$ viewed as a submodule of $\Omega_n(X)$; it thus follows that the inclusions $\Omega_n(A) \hookrightarrow \Omega_n(X)$ and $\Omega_n(A) \hookrightarrow \Omega_n(A) \oplus \widehat{\Omega}_n(X, A)$ are isomorphic. \square

COROLLARY 4.8. Given a cofibration of directed graphs $A \hookrightarrow X$, for $n \geq 0$, the quotient module $\Omega_n(X)/\Omega_n(A)$ is isomorphic to $\widehat{\Omega}_n(X, A)$. Moreover, these isomorphisms define an isomorphism of chain complexes $\Omega(X)/\Omega(A) \cong \widehat{\Omega}(X, A)$.

Proof. The stated isomorphisms of R -modules are immediate from Lemma 4.7. That these isomorphisms commute with the boundary operations of $\Omega(X)/\Omega(A)$ and $\widehat{\Omega}(X, A)$ follows from a routine calculation. \square

REMARK 4.9. Note that the proofs of Lemma 4.7 and Corollary 4.8 do not require the condition that $A \hookrightarrow X$ admits a projecting decomposition, only that there are no edges out of A in X . Indeed, [14, Lem. 3.10] assumes only (the dual of) the latter condition.

DEFINITION 4.10. We define a functor $\widehat{\Omega}: \text{DiGraph}_2 \rightarrow \text{Ch}_R$ as follows. Given a cofibration $A \hookrightarrow X$, its image under $\widehat{\Omega}$ is the chain complex $\widehat{\Omega}(X, A)$. Given a commuting square $f: (X, A) \rightarrow (Y, B)$, each R -module homomorphism $\widehat{\Omega}_n(f): \widehat{\Omega}_n(X, A) \rightarrow \widehat{\Omega}_n(Y, B)$ is the composite of the restriction of $\Omega_n(f): \Omega_n(X) \rightarrow \Omega_n(Y)$ with the quotient map $\Omega_n(Y) \cong \Omega_n(B) \oplus \widehat{\Omega}_n(Y, B) \rightarrow \widehat{\Omega}_n(Y, B)$ which sends the summand $\Omega_n(B)$ to 0.

PROPOSITION 4.11. The functors $Q, \widehat{\Omega}: \text{DiGraph}_2 \rightarrow \text{Ch}_R$ are naturally isomorphic.

Proof. Given a cofibration $X \hookrightarrow A$, its images under Q and $\widehat{\Omega}$ are isomorphic by Corollary 4.8. To see that this isomorphism is natural, consider a morphism $f: (X, A) \rightarrow (Y, B)$ in DiGraph_2 . Given $\omega \in \widehat{\Omega}_n(X, A)$, the composites $\widehat{\Omega}_n(X, A) \cong \Omega_n(X)/\Omega_n(A) \xrightarrow{Qf} \Omega_n(Y)/\Omega_n(B)$ and $\widehat{\Omega}_n(X, A) \xrightarrow{\widehat{\Omega}(f)} \widehat{\Omega}_n(Y, B) \cong \Omega_n(Y)/\Omega_n(B)$ both send ω to the equivalence class of $f(\omega)$ modulo $\Omega_n(B)$ in $\Omega_n(Y)$. \square

We next construct the functor $M: \text{DiGraph}_2^{\text{PO}} \rightarrow \text{Ch}_R$.

DEFINITION 4.12. Let $A \hookrightarrow X$ be a cofibration. For $n \geq 0$, let $\widehat{A}_n^1(X, A)$ denote the submodule of $A_n(X - A)$ generated by the paths whose last vertices are in the subgraph

$(X - A)_1$. The R -module $\widehat{\Omega}_n^1(X, A)$ is defined by the following pullback diagram:

$$\begin{array}{ccc} \widehat{\Omega}_n^1(X, A) & \longrightarrow & \widehat{A}_{n-1}^1(X, A) \\ \downarrow & \lrcorner & \downarrow \\ \widehat{A}_n^1(X, A) & \xrightarrow{\partial} & C_{n-1}(X - A) \end{array}$$

In other words, $\widehat{\Omega}_n^1(X, A)$ is the submodule of $\Omega_n(X)$ consisting of linear combinations of allowed paths with their last vertices in $(X - A)_1$, whose boundaries again have their last vertices in $(X - A)_1$. Similar to the definition of $\widehat{\Omega}(X, A)$, the boundary map on $C_n(X)$ restricts to a map $\widehat{\Omega}_n^1(X, A) \rightarrow \widehat{\Omega}_{n-1}^1(X, A)$ for each n ; we thus obtain a chain complex $\widehat{\Omega}^1(X, A) \subseteq \widehat{\Omega}(X, A)$.

Now consider a morphism in $\text{DiGraph}_2^{\text{PO}}$, i.e. a pushout diagram in which the vertical maps are cofibrations, as depicted below:

$$\begin{array}{ccc} A & \xrightarrow{f|_A} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

By Corollary 2.11, for each n the map $C_n(X) \rightarrow C_n(Y)$ restricts to an isomorphism $C_n(X - A) \cong C_n(Y - B)$, and these in turn restrict to isomorphisms $\widehat{A}_n^1(X, A) \rightarrow \widehat{A}_n^1(Y, B)$ which commute with boundaries. Thus we have a natural isomorphism of cospans:

$$\begin{array}{ccccc} \widehat{A}_{n-1}^1(X, A) & \hookrightarrow & C_{n-1}(X - A) & \xleftarrow{\partial} & \widehat{A}_n^1(X, A) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \widehat{A}_{n-1}^1(Y, B) & \hookrightarrow & C_{n-1}(Y - B) & \xleftarrow{\partial} & \widehat{A}_n^1(Y, B) \end{array}$$

This yields a natural isomorphism between pullbacks $\widehat{\Omega}_n^1(X, A) \cong \widehat{\Omega}_n^1(Y, B)$. As this map commutes with boundaries by construction, it defines a chain complex isomorphism $\widehat{\Omega}^1(X, A) \cong \widehat{\Omega}^1(Y, B)$.

We thus define a functor $\widehat{\Omega}^1: \text{DiGraph}_2^{\text{PO}} \rightarrow \text{Ch}_R^{\rightarrow}$ sending each object $A \mapsto X$ to the chain complex inclusion $\widehat{\Omega}^1(X, A) \hookrightarrow \Omega(X - A)$ and each morphism $(X, A) \rightarrow (Y, B)$ of $\text{DiGraph}_2^{\text{PO}}$ to the commuting square below:

$$\begin{array}{ccc} \widehat{\Omega}^1(X, A) & \xrightarrow{\cong} & \widehat{\Omega}^1(Y, B) \\ \downarrow & & \downarrow \\ \Omega(X - A) & \xrightarrow{\cong} & \Omega(Y - B) \end{array}$$

It is straightforward to verify that this construction is functorial.

4.3. MAPPING CONE COMPLEX OF A COFIBRATION.

DEFINITION 4.13. Let $M: \text{DiGraph}_2^{\text{PO}} \rightarrow \text{Ch}_R$ denote the composite of the functor $\widehat{\Omega}^1$, defined above, with the mapping cone functor $\text{Ch}_R^{\rightarrow} \rightarrow \text{Ch}_R$. We refer to $M(X, A)$ as the mapping cone complex of $A \mapsto X$.

More explicitly, the functor M may be described as follows:

- for $n \geq 0$, $M_n(X, A) = \widehat{\Omega}_{n-1}^1(X, A) \oplus \Omega_n(X - A)$;
- for $(p, q) \in M_n(X, A)$, $\partial(p, q) = (-\partial p, \partial q - p)$.

- given a pushout square $f: (X, A) \rightarrow (X', A')$, Mf acts in each degree n as the direct sum of the isomorphisms $\widehat{\Omega}_{n-1}^1(X, A) \cong \widehat{\Omega}_{n-1}^1(X', A')$ and $\Omega_n(X - A) \cong \Omega_n(X' - A')$ induced by f .

The following result is then immediate.

LEMMA 4.14. *The functor $M: \text{DiGraph}_2^{\text{PO}} \rightarrow \text{Ch}_R$ sends all morphisms of $\text{DiGraph}_2^{\text{PO}}$ to isomorphisms of Ch_R . \square*

Our next goal is to establish a natural isomorphism E from M to the restriction of $\widehat{\Omega}$ to $\text{DiGraph}_2^{\text{PO}}$. (For ease of notation, we also write $\widehat{\Omega}$ for the restriction of $\widehat{\Omega}: \text{DiGraph}_2 \rightarrow \text{Ch}_R$ to $\text{DiGraph}_2^{\text{PO}}$, relying on context to remove ambiguity.)

The definition of E is fairly involved and will only be given in Definition 4.24, since its well-definedness (Proposition 4.25) depends on preceding results. To provide a roadmap, let us briefly summarize the strategy. We begin by defining linear maps L^j acting on chain complexes associated to the cofibration $A \twoheadrightarrow X$. Although at that point we could state the definition of E , since the required formula only involves the function L^0 , we defer the definition, by first proving a sequence of lemmas (Lemmas 4.16 - 4.23) that establish well-definedness of E .

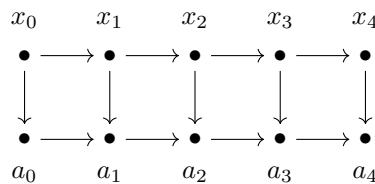
From here on, let $A \twoheadrightarrow X$ denote an arbitrary cofibration. We will regularly write an arbitrary path in X as $x = x_0 \cdots x_n$ and will write a_i for the vertex $\pi x_i \in A_V$.

DEFINITION 4.15. *For $n \geq 1$, we define a family of linear maps $L^j: C_{n-1}((X - A)^A) \rightarrow C_n(X)$ for $0 \leq j \leq n - 1$. Given a generator $x = x_0 \cdots x_{n-1} \in C_{n-1}((X - A)^A)$, let*

$$L^j(x) = \sum_{i=j}^{n-1} (-1)^i x_0 \cdots x_i a_i \cdots a_{n-1}.$$

This definition extends by linearity to define each L^j on all of $C_{n-1}((X - A)^A)$.

As the definition above may appear very technical, we will briefly discuss some of the intuition behind it. Suppose we are given a path $x = x_0 \cdots x_{n-1}$ lying entirely in $(X - A)_1$, i.e. such that $h(x_i) = 1$ for all i . Then by Lemma 3.5 and Lemma 3.8, the graph A contains a grid of squares formed by the path x , its projection to A , and the edges $x_i \rightarrow a_i$, pictured below for a path of length 4.



The element $L^0(x) \in C_n(X)$ is the alternating sum of all paths from the upper left corner (i.e. x_0) to the lower right corner (i.e. a_4) of this grid. It can be seen as a generalization of the generator of $\Omega_2(C_{2,2})$ of Example 2.26. Our strategy for proving the desired isomorphism, roughly speaking, involves showing that the cofibration conditions are sufficiently restrictive that any path in X which forms part of a linear combination in $\widehat{\Omega}_n(X, A)$ must either be contained entirely in $(X - A)$, or arise from a grid construction similar to the above (suitably generalized for paths not contained entirely in $(X - A)_1$).

LEMMA 4.16. *For $n \geq 1$ and $0 \leq j \leq n - 1$, the map $L^j: C_{n-1}((X - A)^A) \rightarrow C_n(X)$ is injective.*

Proof. We first prove this result in the case $j = n - 1$. Here we may note that for any generator (i.e. any non-degenerate path) $x = x_0 \cdots x_{n-1}$ of $C_{n-1}((X - A)^A)$, we have $L^{n-1}(x) = (-1)^{n-1}x_0 \cdots x_{n-1}a_{n-1}$. This is necessarily non-degenerate: the assumption that x is non-degenerate implies $x_i \neq x_{i+1}$ for $0 \leq i \leq n - 2$, and $x_{n-1} \neq a_{n-1}$ since x_{n-1} is not in A . Taking as a basis for $C_n(X)$ the set of elements $(-1)^{n-1}y$ for all non-degenerate n -paths y in X , we thus see that distinct generators of $C_{n-1}((X - A)^A)$ are sent to distinct elements of this basis. It follows that L^{n-1} is injective.

Now consider an arbitrary $0 \leq j \leq n - 2$. Suppose that for some $p \in C_{n-1}((X - A)^A)$ we have $L^j(p) = 0$. We may rewrite this equation as $L^j(p) - L^{n-1}(p) = -L^{n-1}(p)$. Now note that for any generator x as above, we have $L^j(x) - L^{n-1}(x) = \sum_{i=j}^{n-2} (-1)^i x_0 \cdots x_i a_i \cdots a_{n-1}$. Thus each nonzero term of $L^j(p) - L^{n-1}(p)$ corresponds to a path including at least two vertices of A , while each nonzero term of $L^{n-1}(p)$ corresponds to a path including exactly one vertex of A . Thus this equality can only hold if both sides are equal to zero. In particular, we have $L^{n-1}(p) = 0$; as shown above, this implies $p = 0$. \square

LEMMA 4.17. For $n \geq 2$, let $x = x_0 \cdots x_{n-1}$ denote a generator of $\widehat{A}_{n-1}^1(X, A)$. Suppose that for some $0 \leq j \leq n - 2$, we have $a_j = a_{j+1}$. Then $L^{j'}(x) = L^0(x)$ for all $j' \leq j$.

Proof. By definition, $L^0(x) = \sum_{i=0}^{n-1} (-1)^i x_0 \cdots x_i a_i \cdots a_{n-1}$. Thus we must show that all terms of this sum for which $i < j$ are zero. This follows from the fact that each such term includes the substring $a_j a_{j+1}$, and is therefore degenerate. \square

LEMMA 4.18. For $n \geq 2$, let $x = x_0 \cdots x_{n-1}$ denote a generator of $\widehat{A}_{n-1}^1(X, A)$. Let $0 \leq j \leq n - 1$ be minimal such that $h(x_i) = 1$ for all $i > j$. Then $L^0(x) = L^j(x)$.

Proof. If $j = 0$ then the statement is a tautology, so assume otherwise. Furthermore, we may note that because x is an allowed path with its last vertex in $(X - A)_1$, we have $h(x_{n-1}) = 1$, so that $j < n - 1$.

By the minimality of j , we have $h(x_{j-1}) \neq 1$; as x_{j-1} is not in A , it follows that $h(x_{j-1}) \geq 2$. As there is an edge $x_{j-1} \rightarrow x_j$, Lemma 3.8 implies that $a_{j-1} = a_j$. The stated result thus follows from Lemma 4.17. \square

LEMMA 4.19. For all $n \geq 1$ and $p \in C_{n-1}((X - A)^A)$, we have $L^0(p) \in \widehat{A}_n(X, A)$ if and only if $p \in \widehat{A}_{n-1}^1(X, A)$.

Proof. We first show that if $p \in \widehat{A}_{n-1}^1(X, A)$ then $L^0(p) \in \widehat{A}_n(X, A)$. It suffices to show that, for all generators $x \in \widehat{A}_n^1(X, A)$, $L^0(x)$ is allowed and intersects $X - A$. By Lemma 4.18, to show that $L^0(x)$ is allowed, it suffices to show that each path $x_1 \cdots x_i a_i \cdots a_{n-1}$ is allowed for $i \geq j$, where j is minimal such that $h(x_i) = 1$ for all $i \geq j$. Therefore, let $x = x_0 \cdots x_{n-1}$ be an allowed path in X with its last vertex in A , and consider the path

$$x_0 \cdots x_i a_i \cdots a_{n-1}$$

for some i satisfying $j \leq i \leq n - 1$. For each such $0 \leq k \leq i - 1$, there is an edge $x_k \rightarrow x_{k+1}$ by the assumption that p is allowed. Similarly, for $i \leq k < n - 1$, the assumption that p is allowed implies that there is an edge $x_k \rightarrow x_{k+1}$. Furthermore, the assumption that $j \leq i$ implies that $h(x_k) = h(x_{k+1}) = 1$; these two facts imply the existence of an edge $a_k \rightarrow a_{k+1}$ by Lemma 3.8. It remains to be shown that there is an edge $x_i \rightarrow a_i$; this is immediate from the assumption that $h(x_i) = 1$ and Lemma 3.5.

Now suppose that $L^0(p) \in \widehat{A}_n(X, A)$. We may write p as a linear combination of generators $\sum_{k=1}^m c_k x^k$, where each c_k is a coefficient and each x^k is an allowed path in $(X - A)^A$. Then the sum of all terms of $L^0(p)$ corresponding to paths which include exactly one vertex of A is

$$L^{n-1}(p) = \sum_{k=1}^m c_k (-1)^{n-1} x_0^k \cdots x_{n-1}^k a_{n-1}^k$$

where $a_{n-1}^k = \pi x_{n-1}^k$. Since the generators x^k are distinct, the terms of this sum are distinct as well. Thus there are no cancellations among these terms. As all remaining terms of $L^0(p)$ correspond to paths including at least two vertices of A , none of these terms can cancel with those of the sum above either. Therefore, since $L^0(p)$ is allowed, it must be the case that each path $x_0^k \cdots x_{n-1}^k a_{n-1}^k$ is allowed. In particular, this implies that $x_0^k \cdots x_{n-1}^k$ is allowed. Furthermore, the presence of an edge $x_{n-1}^k \rightarrow a_{n-1}^k$ implies that $h(x_{n-1}^k) = 1$. Thus $p \in \widehat{A}_{n-1}^1(X, A)$. \square

DEFINITION 4.20. For $n \geq 0$, we define a linear map $\pi: C_n((X - A)^A) \rightarrow C_n(A)$ by sending each generator $x = x_0 \cdots x_n$ to $a_0 \cdots a_n$.

LEMMA 4.21. If $p \in \widehat{A}_n^1(X, A)$, then $\pi p \in A_n(A)$.

Proof. It suffices to consider the case where p is a generator, i.e. an allowed path $x = x_0 \cdots x_n$; in this case we have $\pi x = a_0 \cdots a_n$. Our assumption that $x \in \widehat{A}_n^1(X, A)$ implies $h(x_n) = 1$, and that $h(x_i) \geq 1$ for all i , since x_n is not in A_V and there are no edges out of A .

If $h(x_i) = 1$ for all i , then πx is allowed by Lemma 3.8.(1). Now suppose that $h(x_j) > 1$ for some j ; without loss of generality we may choose j to be maximal, so that $h(x_{j+1}) = 1$. Then by Lemma 3.8.(2), we have $a_j = a_{j+1}$. Thus πx is degenerate, and hence equal to zero. \square

LEMMA 4.22. For $n \geq 1$ and $p \in \widehat{A}_{n-1}^1(X, A)$, we have $\partial L^0(p) = -L^0(\partial p) - p + \pi(p)$.

Proof. It suffices to show the given equality when p is a generator $x = x_0 \cdots x_{n-1}$. In this case we compute:

$$\begin{aligned} \partial L^0(x) &= \partial \left(\sum_{i=0}^{n-1} (-1)^i x_0 \cdots x_i a_i \cdots a_{n-1} \right) \\ &= \sum_{i=0}^{n-1} (-1)^i \partial x_0 \cdots x_i a_i \cdots a_{n-1} \\ &= \sum_{i=0}^{n-1} (-1)^i \left(\sum_{j=0}^i (-1)^j x_0 \cdots \widehat{x}_j \cdots x_i a_i \cdots a_{n-1} \right. \\ &\quad \left. + \sum_{j=i}^{n-1} (-1)^{j+1} x_0 \cdots x_i a_i \cdots \widehat{a}_j \cdots a_{n-1} \right) \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^i (-1)^{i+j} x_0 \cdots \widehat{x}_j \cdots x_i a_i \cdots a_{n-1} \\ &\quad + \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{i+j+1} x_0 \cdots x_i a_i \cdots \widehat{a}_j \cdots a_{n-1}. \end{aligned}$$

At this point, we may note that for each $1 \leq i \leq n-1$, the (i, i) -term of the left summation is $x_0 \cdots x_{i-1} a_i \cdots a_{n-1}$, while the $(i-1, i-1)$ -term of the right summation is $-x_0 \cdots x_{i-1} a_i \cdots a_{n-1}$. Thus these terms cancel. Furthermore, the $(0, 0)$ -term of the left summation is $a_0 \cdots a_{n-1} = \pi x$, and the $(n-1, n-1)$ -term of the right summation is equal to $-x_0 \cdots x_{n-1} = -x$.

For the sake of readability, we will set these terms aside and show that the remaining part of the sum is equal to $-L^0(\partial x)$. We are thus left to consider:

$$\sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-1)^{i+j} x_0 \cdots \hat{x}_j \cdots x_i a_i \cdots a_{n-1} + \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} (-1)^{i+j+1} x_0 \cdots x_i a_i \cdots \hat{a}_j \cdots a_{n-1}.$$

We reverse the order of summation in both terms, summing first over j , then over i . Thus we obtain:

$$\begin{aligned} & \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} (-1)^{i+j} x_0 \cdots \hat{x}_j \cdots x_i a_i \cdots a_{n-1} + \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} (-1)^{i+j+1} x_0 \cdots x_i a_i \cdots \hat{a}_j \cdots a_{n-1} \\ &= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (-1)^{i+j} x_0 \cdots \hat{x}_j \cdots x_i a_i \cdots a_{n-1} + \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} (-1)^{i+j+1} x_0 \cdots x_i a_i \cdots \hat{a}_j \cdots a_{n-1} \\ &= \sum_{j=0}^{n-1} (-1)^j \left(\sum_{i=j+1}^{n-1} (-1)^i x_0 \cdots \hat{x}_j \cdots x_i a_i \cdots a_{n-1} + \sum_{i=0}^{j-1} (-1)^{i+1} x_0 \cdots x_i a_i \cdots \hat{a}_j \cdots a_{n-1} \right). \end{aligned}$$

For each $0 \leq j \leq n-1$, let $y^j = \partial_j x$. That is, for $0 \leq i < j$, we have $y_i^j = x_i$, while for $j \leq i \leq n-2$ we have $y_i^j = x_{i+1}$. For each j, i let $b_i^j = \pi y_i^j$. Then we may rewrite the expression above as:

$$\begin{aligned} & \sum_{j=0}^{n-1} (-1)^j \left(\sum_{i=j+1}^{n-1} (-1)^i y_0^j \cdots y_{i-1}^j b_{i-1}^j \cdots b_{n-2}^j + \sum_{i=0}^{j-1} (-1)^{i+1} y_0^j \cdots y_i^j b_i^j \cdots b_{n-2}^j \right) \\ &= \sum_{j=0}^{n-1} (-1)^j \left(\sum_{i=j}^{n-2} (-1)^{i+1} y_0^j \cdots y_i^j b_i^j \cdots b_{n-2}^j + \sum_{i=0}^{j-1} (-1)^{i+1} y_0^j \cdots y_i^j b_i^j \cdots b_{n-2}^j \right) \\ &= \sum_{j=0}^{n-1} (-1)^j \sum_{i=0}^{n-2} (-1)^{i+1} y_0^j \cdots y_i^j b_i^j \cdots b_{n-2}^j \\ &= \sum_{j=0}^{n-1} (-1)^j (-L^0(y^j)) \\ &= -L^0 \left(\sum_{j=0}^{n-1} (-1)^j y^j \right) \\ &= -L^0(\partial x). \end{aligned}$$

Thus the statement is proven. \square

LEMMA 4.23. For $n \geq 1$, if $p \in \widehat{\Omega}_{n-1}^1(X, A)$, then $L^0(p) \in \widehat{\Omega}_n(X, A)$.

Proof. From Lemma 4.19, we have that $L^0(p) \in \widehat{A}_n(X, A)$. To see that $\partial L^0(p)$ is allowed, recall that by Lemma 4.22, $\partial L^0(p) = -L^0(\partial p) - p + \pi p$. We consider each of these terms individually.

- ∂p is allowed by assumption, hence $L^0(\partial p)$ is allowed by Lemma 4.19.

- p is allowed by assumption.
- πp is allowed by Lemma 4.21. □

From Lemma 4.23, it follows that L^0 restricts to define a linear map

$$\widehat{\Omega}_{n-1}^1(X, A) \rightarrow \widehat{\Omega}_n(X, A).$$

DEFINITION 4.24. For $n \geq 0$, we define a linear map $E: M_n(X, A) \rightarrow \widehat{\Omega}_n(X, A)$, sending an element $(p, q) \in M_n(X, A) = \widehat{\Omega}_{n-1}^1(X, A) \oplus \Omega_n(X - A)$ to $L^0(p) + q$.

PROPOSITION 4.25. The maps $E: M_n(X, A) \rightarrow \widehat{\Omega}_n(X, A)$ of Definition 4.24 define a map of chain complexes $E: M(X, A) \rightarrow \widehat{\Omega}(X, A)$. Moreover, these maps define a natural transformation $E: M \Rightarrow \widehat{\Omega}$.

Proof. Consider an arbitrary element $(p, q) \in M_n(X, A) = \widehat{\Omega}_{n-1}^1(X, A) \oplus \widehat{\Omega}_n(X - A)$. We will show that $\partial E(p, q) = E(\partial(p, q))$.

First, consider $\partial E(p, q) = \partial(L^0(p) + q)$. Applying Lemma 4.22, and recalling that terms corresponding to paths contained entirely in A are set to zero when computing boundaries in $\widehat{\Omega}(X, A)$, this is equal to $-L^0(\partial p) - p + \partial q$. Now consider $E(\partial(p, q))$; by definition, this is $E(-\partial p, \partial q - p) = -L^0(\partial p) + \partial q - p$. Thus we see that the two terms are equal.

To prove the naturality of E , we must show that the following square commutes, for any pushout square $f: (X, A) \rightarrow (Y, B)$.

$$\begin{array}{ccc} M(X, A) & \xrightarrow{E} & \widehat{\Omega}(X, A) \\ \downarrow & & \downarrow \\ M(Y, B) & \xrightarrow{E} & \widehat{\Omega}(Y, B) \end{array}$$

This follows by a straightforward computation. □

PROOF OF EXCISION AXIOM. Our next goal will be to prove that $E: M \Rightarrow \widehat{\Omega}$ is a natural isomorphism, from which the proof of Theorem 4.5 will follow. For this, we will require some further lemmas characterizing the elements of $\widehat{\Omega}_n(X, A)$. As the proofs of these lemmas are very long and technical, we will first discuss a simple example to illustrate some of the essential ideas behind them.

Suppose that an element $\omega \in \widehat{\Omega}_3(X, A)$, viewed as a sum of non-degenerate allowed paths, contains a term of the form $cx_0x_1a_1a_2$, where c is a non-zero element of \mathcal{R} , $x_0, x_1 \in (X - A)_V$ and $a_1, a_2 \in A_V$. For brevity, we let z denote the path $x_0x_1a_1a_2$.

Consider the boundary of z : its 2-face, in particular, is $cx_0x_1a_2$. The assumption that the original path was non-degenerate implies that $a_1 \neq a_2$. Applying Lemma 3.5, we see that $\pi x_1 = a_1$ (because there is an edge $x_1 \rightarrow a_1$), but that because $a_1 \neq a_2$, there is no edge $x_1 \rightarrow a_2$. Thus $x_0x_1a_2$ is not allowed.

By assumption, the boundary of ω is allowed, so there must be some other terms of $\partial\omega$, arising from the boundaries of other terms of ω , which will cancel this one. Thus the linear combination ω must contain some sum $\sum_{i=1}^m c_i z_i$ such that we can obtain $x_0x_1a_2$ from each path z_i by omitting a vertex. Because all terms of ω correspond to allowed paths, and there is no edge $x_1 \rightarrow a_2$, for each z_i the vertex to be omitted must appear in between x_1 and a_2 , i.e. we must have $z_i = x_0x_1v_ia_2$ for some vertex v_i . Thus $x_0x_1a_2$ is again the 2-face of each z_i , and hence appears in their boundaries with positive sign; it follows that $\sum_{i=1}^m c_i = -c$, so that these terms, when added to $cx_0x_1a_2$, will give 0. We may assume that we are working with a non-redundant presentation of ω , so that the z_i and z are all distinct; in particular, this implies $v_i \neq a_1$ for all i as z and z_i can differ only in this vertex. By Lemma 3.5, it follows

that no v_i is a vertex of A , as each one admits an edge from x_1 , and the only vertex of A admitting an edge from x_1 is a_1 . Thus each v_i is a vertex of $X - A$ for which there exist edges $x_1 \rightarrow v_i$ and $v_i \rightarrow a_2$. Applying Lemma 3.5 again, it follows that $\pi v_i = a_2$ for all i . If we then let $x_2^i = v_i$ for all i , we can rearrange the sum $cz + \sum_{i=1}^m c_i z_i$ as follows:

$$\begin{aligned} cz + \sum_{i=1}^m c_i z_i &= cx_0 x_1 a_1 a_2 + \sum_{i=1}^m c_i z_i \\ &= - \sum_{i=1}^m c_i x_0 x_1 a_1 a_2 + \sum_{i=1}^m c_i x_0 x_1 x_2^i a_2 \\ &= \sum_{i=1}^m (-c_i x_0 x_1 a_1 a_2 + c_i x_0 x_1 x_2^i a_2) \\ &= \sum_{i=1}^m c_i L^1(x_0 x_1 x_2^i) \\ &= cL^1(x_0 x_1 x_2^i). \end{aligned}$$

Thus z appears in the linear combination ω as part of an ‘ L -term,’ that is, a term in the image of some L^j .

In Lemma 4.26, we will generalize the reasoning of this example to show that the terms of any element of $\widehat{\Omega}_n(X, A)$ which correspond to paths intersecting A can be grouped into L -terms. Then, in Lemma 4.27, we will apply similar reasoning to show that any such element may be expressed as a sum of terms in $\Omega_n(X - A)$ and terms in the image of L^0 .

LEMMA 4.26. *For $n \geq 0$, every element $\omega \in \widehat{\Omega}_n(X, A)$ can be written as*

$$\omega = q + \sum_{k=1}^m c_k L^{j_k}(x^k)$$

for some $q \in A_n(X - A)$ and some set of indices $0 \leq k \leq m$ and $0 \leq j_k \leq n - 1$, where $x^k = x_0^k \cdots x_{n-1}^k$ are distinct non-degenerate allowed paths satisfying $h(x_{n-1}^k) = 1$ and $c_k \in R$ are nonzero.

Proof. We first note that any element $\omega \in \widehat{\Omega}_n(X, A)$ may be expressed as

$$(1) \quad \omega = q + \sum_{k=1}^m c_k L^{j_k}(x^k) + \sum_{r=1}^s d_r y^r$$

where:

- $q + \sum_{k=1}^m c_k L^{j_k}(x^k)$ satisfies the conditions given in the statement;
- $s \geq 0$, each d_r is a nonzero element of R and each y^r is a distinct allowed path in X intersecting both $X - A$ and A .

To obtain an expression as in Eq. (1) for an arbitrary element ω , we group its terms which do not intersect A together as q , and take the sum of the remaining terms to be $\sum_{r=1}^s d_r y^r$, setting $m = 0$. Furthermore, we may assume without loss of generality that our chosen presentation of ω is non-redundant, i.e. that the terms y^r and those of q all represent distinct non-degenerate paths.

The form of Eq. (1) essentially represents an intermediate state between an arbitrary expression for ω and an expression of the form given in the statement of the lemma. The sum $\sum_{k=1}^m c_k L^{j_k}(x^k)$ consists of those terms of ω which have been grouped together into L -terms as required by the statement, while $\sum_{r=1}^s d_r y^r$ consists

of those terms intersecting A which remain ungrouped. Though we will proceed by a multi-stage induction over several variables, the core of our approach will be to group the terms $d_r y^r$ together into L -terms.

Given an element $\omega \in \widehat{\Omega}_{n-1}(X, A)$ expressed as in Eq. (1), we will show by induction on s that ω may be expressed in the form given in the statement. The base case $s = 0$ is trivial, as this is precisely the case in which the two forms coincide.

Now let $s \geq 1$ and suppose the result is proven for all $0 \leq s' < s$. Choose an arbitrary term $d_r y^r$; for ease of notation we will rename d_r to e and y^r to z . Note that because ez is not a term of q , some vertex of z must be contained in A_V . Because there are no arrows out of A , there exists $0 \leq j \leq n - 1$ such that $z_i \in (X - A)_V$ for $0 \leq i \leq j$, while $z_i \in A_V$ for $j + 1 \leq i \leq n$. Write $z = z_0 \cdots z_j b_j \cdots b_{n-1}$ for $z_i \in (X - A)_V$ and $b_i \in A_V$. This path is allowed, so by Lemma 3.5 we conclude that $h(z_j) = 1$ and $\pi z_j = b_j$.

We now show that for any t with $j \leq t \leq n - 1$, ω may be expressed as

$$q + \sum_{k=1}^m c_k L^{jk}(x^k) + \sum_{r'=1}^{s'} d_{r'} y^{r'} + \sum_{l=1}^u \sum_{i=j}^t (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l$$

for some $s' < s$ and some family of paths $z_0^l \cdots z_t^l$ and coefficients $e^l \in R$ indexed by $1 \leq l \leq u$ for some $u \geq 1$, such that $h(z_i^l) = 1$ for all i between j and t inclusive, where b_i^l denotes πz_i^l for each (l, i) . We assume that the terms of the double summation are distinct from those of the other summands in this presentation of ω , but not necessarily from each other, i.e. we may have $z^l = z^{l'}$ for some l, l' . In the case $t = n - 1$ the double summation will simply become a sum of L -terms, allowing us to apply the induction hypothesis on s to conclude the overall proof.

We proceed by induction on t . In the base case $t = j$, we set $u = 1$, so that the given double sum is simply a single term $e^1 z_0^1 \cdots z_j^1 b_j^1 \cdots b_{n-1}^1$. To express ω in this form we may separate out the chosen term ez , designate it as $e^1 z^1$, set $s' = s - 1$, and re-index the remaining terms.

Now suppose the statement holds for some $j \leq t \leq n - 2$; we will prove it for $t + 1$. For each l , consider the final term of the corresponding alternating sum, $(-1)^{j-t} e^l z_0^l \cdots z_t^l b_t^l \cdots b_{n-1}^l$. The boundary of this term contains a term of the form $(-1)^{j+1} e^l z_0^l \cdots z_t^l b_{t+1}^l \cdots b_{n-1}^l$, obtained by omitting the $(t + 1)^{\text{st}}$ vertex of the path. Our assumption that the path $z_0^l \cdots z_t^l b_{t+1}^l \cdots b_{n-1}^l$ is non-degenerate implies that $b_t^l \neq b_{t+1}^l$, and hence that $\pi z_t^l \neq b_{t+1}^l$. By Lemma 3.5, it follows that there is no edge from z_t^l to b_{t+1}^l ; thus this path is not allowed.

Because the boundary of ω is allowed, this term must therefore be cancelled by some set of other terms of $\partial\omega$. In other words, the linear combination $\partial\omega$ must contain some sum $\sum_w r_w z_0^l \cdots z_t^l b_{t+1}^l \cdots b_{n-1}^l$, where $\sum_w r_w = (-1)^j e^l$, with each term $r_w z_0^l \cdots z_t^l b_{t+1}^l \cdots b_{n-1}^l$ arising as a face of a term of either q , $\sum_{k=1}^m c_k L^{jk}(x^k)$, $\sum_{r'=1}^{s'} d_{r'} y^{r'}$, or $\sum_{l=1}^u \sum_{i=j}^t (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l$. Because the paths z^l are not assumed to be distinct, if there is more than one such term we may re-express the sum indexed by l as follows:

$$\sum_{i=j}^t (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l = - \sum_w \sum_{i=j}^t (-1)^{j-i} r_w z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l$$

Thus we obtain a new expression for ω in the given form for a larger value of u . We may therefore assume without loss of generality that $(-1)^{j+1} e^l z_0^l \cdots z_t^l b_{t+1}^l \cdots b_{n-1}^l$ is canceled by a single term $(-1)^j e^l z_0^l \cdots z_t^l b_{t+1}^l \cdots b_{n-1}^l$ arising in one of the four ways described above.

We may note that, because there is no edge $z_t^l \rightarrow b_{t+1}^l$, the only way in which $(-1)^j e^l z_0^l \cdots z_t^l b_{t+1}^l \cdots b_{n-1}^l$ can arise by omitting a vertex of an allowed path is if the vertex to be omitted appears between z_t^l and b_{t+1}^l . Thus our given presentation of ω contains a term $(-1)^{j-t-1} e^l z_0^l \cdots z_t^l v b_{t+1}^l \cdots b_{n-1}^l$ from which this boundary term arises. We first note that this term cannot be part of q , as the vertices b_i^l are all contained in A .

Next we consider the case in which $(-1)^{j-t-1} e^l z_0^l \cdots z_t^l v b_{t+1}^l \cdots b_{n-1}^l$ is a term of the double summation $\sum_{l=1}^u \sum_{i=j}^t (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l$, corresponding to some index l' (necessarily distinct from l as e^l appears here with opposite sign). Then we must have $v = \pi z_t^l = b_t^l$ by construction of the double sum; it follows that $z^l = z^{l'}, e^l = -e^{l'}$, so that the two terms cancel; by removing these terms and re-indexing for a smaller value of u , we may assume without loss of generality that this case does not occur.

It remains to consider the cases in which $(-1)^{j-t-1} e^l z_0^l \cdots z_t^l v b_{t+1}^l \cdots b_{n-1}^l$ is a term of $\sum_{k=1}^m c_k L^{j_k}(x^k)$ or $\sum_{r'=1}^{s'} d_{r'} y^{r'}$. As the terms of both of these sums are assumed to correspond to paths distinct from those of the double summation, we cannot have $v = b_t^l$ in these cases. As there is an edge $z_t^l \rightarrow v$, we must therefore have v equal to some vertex z_{t+1}^l of $X - A$ admitting an edge from z_t^l and an edge to b_{t+1}^l . In particular, this implies $h(z_{t+1}^l) = 1$ and $\pi z_{t+1}^l = b_{t+1}^l$.

Of these two, we first consider the case in which $(-1)^{j-t-1} e^l z_0^l \cdots z_{t+1}^l b_{t+1}^l \cdots b_{n-1}^l$ is a term of $\sum_{r'=1}^{s'} d_{r'} y^{r'}$. In this case we may simply group the term

$$(-1)^{j-t-1} e^l z_0^l \cdots z_{t+1}^l b_{t+1}^l \cdots b_{n-1}^l$$

together with the summation $\sum_{i=j}^t (-1)^{j-i} e^l y_0 \cdots y_i b_i \cdots b_{n-1}$ to form:

$$\begin{aligned} & (-1)^{j-t-1} e^l z_0^l \cdots z_{t+1}^l b_{t+1}^l \cdots b_{n-1}^l + \sum_{i=j}^t (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l \\ &= \sum_{i=j}^{t+1} (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l \end{aligned}$$

Thus we have extended the sum with index l by adding a suitable $(t + 1)$ -term.

Finally, we consider the case where $(-1)^{j-t-1} e^l z_0^l \cdots z_{t+1}^l b_{t+1}^l \cdots b_{n-1}^l$ is a term of

$$c_k L^{j_k}(x^k) = \sum_{i=j_k}^{n-1} (-1)^i c_k x_0 \cdots x_i a_i \cdots a_{n-1}$$

for some k ; it is then necessarily the $(t + 1)$ -term of this sum, because $z_{t+1}^l \in (X - A)_V$ while $b_{t+1}^l \in A_V$. Note that for $i \geq j_k + 1$, the i -term of such a sum has as its i^{th} face $c_k x_0 \cdots x_{i-1} a_i \cdots a_{n-1}$, while the $(i - 1)$ -term has as its i^{th} face $-c_k x_0 \cdots x_{i-1} a_i \cdots a_{n-1}$. Thus these two faces will cancel each other. As we are seeking the term of ω whose $(t + 1)^{\text{st}}$ face will cancel that of the $(-1)^{j-t} e^l z_0^l \cdots z_t^l b_t^l \cdots b_{n-1}^l$ term which we identified earlier, we may therefore assume without loss of generality that $(-1)^{j-t-1} z_0^l \cdots z_{t+1}^l b_{t+1}^l \cdots b_{n-1}^l$ is the j_k -term of the sum, i.e. that $j_k = t + 1$ and $c_k = (-1)^j e^l$. Thus we may perform the following rearrangement to remove this term from $c_k L^{t+1}(x^k)$ and group it with $\sum_{i=j}^t (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l$:

$$\begin{aligned}
 & (-1)^j e^l L^{t+1}(x^k) + \sum_{i=j}^t e^l (-1)^{j-i} z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l \\
 &= \sum_{i=t+1}^{n-1} (-1)^{i+j} e^l x_0 \cdots x_i a_i \cdots a_{n-1} + \sum_{i=j}^t (-1)^{j-i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l \\
 &= \sum_{i=t+2}^{n-1} (-1)^{i+j} e^l x_0 \cdots x_i a_i \cdots a_{n-1} + (-1)^{t+1+j} e^l z_0^l \cdots z_{t+1}^l b_{t+1}^l \cdots b_{n-1}^l \\
 &\quad + \sum_{i=j}^t (-1)^{j+i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l \\
 &= \sum_{i=t+2}^{n-1} (-1)^{i+j} e^l x_0 \cdots x_i a_i \cdots a_{n-1} + \sum_{i=j}^{t+1} (-1)^{j+i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l \\
 &= (-1)^j e^l L^{t+2}(x^k) + \sum_{i=j}^{t+1} (-1)^{j+i} e^l z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l.
 \end{aligned}$$

Therefore, in this case as well, we have extended the summation with index l by adding a suitable $(t + 1)$ -term.

Thus ω may be expressed as a sum of the given form for i ranging from j to t for any $j \leq t \leq n - 1$. Considering this result in the case $t = n - 1$, we note that $\sum_{l=1}^u \sum_{i=j}^{n-1} (-1)^{j+i} z_0^l \cdots z_i^l b_i^l \cdots b_{n-1}^l = \sum_{l=1}^u L^j(z_0^l \cdots z_{n-1}^l)$. Moreover, if $z^l = z^{l'}$ for any distinct l, l' , we may at this point sum the corresponding L -terms to obtain a single term with coefficient $e^l + e^{l'}$; thus we may now assume that the paths z^l are all distinct from each other, as well as from the x^k . Thus we may group this term together with the sum $\sum_{k=1}^m c_k L^{j_k}(x^k)$. After suitable re-indexing, we thus obtain an expression for ω of the form

$$q + \sum_{k=1}^m c_k L^{j_k}(x^k) + \sum_{r'=1}^{s'} d_{r'} y^{r'}$$

where $s' < s$. Applying induction on s , we see that

$$\omega = q + \sum_{k=1}^m c_k L^{j_k}(x^k)$$

for a suitable choice of indices. Thus any $\omega \in \widehat{\Omega}_n(X, A)$ may be expressed in the form given in the statement. \square

LEMMA 4.27. For $n \geq 0$, every element $\omega \in \widehat{\Omega}_n(X, A)$ may be written as

$$\omega = q + L^0(p)$$

for some $p \in \widehat{A}_{n-1}^1(X, A)$, $q \in A_n(X - A)$.

Proof. We begin by expressing an arbitrary element ω in the form given by Lemma 4.26:

$$\omega = q + \sum_{k=1}^m c_k L^{j_k}(x^k)$$

We will assume that each lower limit j_k is minimal, meaning that there is no $j'_k < j_k$ such that $L^{j'_k}(x^k) = L^{j_k}(x^k)$. Let m denote the number of lower limits j_k which are

nonzero. We will prove the statement by induction on m . By linearity of L^0 , it suffices to show that ω can be expressed in the form of Lemma 4.26 with all $j_k = 0$.

The base case $m = 0$ is trivial. For the inductive step, we let $m \geq 1$ and suppose the statement is proven for all $m' < m$. Choose some arbitrary k for which $j_k \geq 1$, and consider the term $c_k L^{j_k}(x^k)$. For ease of notation, we let $c_k = c$, $x = x^k$ and $j = j_k$.

The j -term of the sum $cL^j(x) = \sum_{i=j}^{n-1} (-1)^i c x_0 \cdots x_i a_i \cdots a_{n-1}$ is given by $(-1)^j c x_0 \cdots x_j a_j \cdots a_{n-1}$. The j -face of this term is $c x_0 \cdots x_{j-1} a_j \cdots a_{n-1}$. Recall that by assumption, $\pi x_{j-1} \neq a_j$; Lemma 3.5 thus implies that this path is not allowed. Similarly to the proof of Lemma 4.26, the fact that the boundary of ω is allowed implies that some set of other terms of $\partial\omega$ must combine to cancel $c x_0 \cdots x_{j-1} a_j \cdots a_{n-1}$. Once again, the fact that there is no edge $x_{j-1} \rightarrow a_j$ and all terms of ω are allowed implies that each of these terms is of the form $r_w x_0 \cdots x_{j-1} v_w a_j \cdots a_{n-1}$ for some vertex v_w of X , and $\sum_w r_w = -c$. Likewise, we may break up the term $cL^j(x)$ as a sum $-\sum_w r_w L^j(x)$ and consider each of these L -terms individually. We will also assume, without loss of generality, that no subset of the r_w adds to 0 – otherwise we may simply consider the complement of this subset to obtain a smaller set of cancelling terms.

First we consider the case in which v_w is a vertex of A ; then Lemma 3.5 implies that $\pi x_{j-1} = v_w$, so let $a_{j-1} = v_w$. Then $r_w x_0 \cdots x_{j-1} a_{j-1} a_j \cdots a_{n-1}$ is necessarily the $(j - 1)$ -term of $c_{k'} L^{j_{k'}}(x^{k'})$. We note that the j -term of $L^{j_{k'}}(x^{k'})$ is $(-1)^j x_0 \cdots x_{j-1} x'_j a_j \cdots a_{n-1}$, for some vertex x'_j of $X - A$. Thus the j -face of this term is $x_0 \cdots x_{j-1} a_j \cdots a_{n-1}$, which cancels with the j -face of $(-1)^{j+1} x_0 \cdots x_{j-1} a_{j-1} a_j \cdots a_{n-1}$. As $r_w x_0 \cdots x_{j-1} v_w a_j \cdots a_{n-1}$ is meant to be the term whose j -face cancels that of $-r_w x_0 \cdots x_{j-1} x_j a_j \cdots a_{n-1}$ which we initially identified, we may therefore assume without loss of generality that v_w is not a vertex of A .

Let us therefore assume that v_w is a vertex of $X - A$, and let $x'_j = v_w$; then Lemma 3.5 implies that $\pi x'_j = a_j$. The term $r_w x_0 \cdots x_{j-1} x'_j a_j \cdots a_{n-1}$ is necessarily the j -term of $c_{k'} L^{j_{k'}}(x^{k'})$, implying $r_w = (-1)^j c_{k'}$.

If $j_{k'} < j$, then the $(j - 1)$ -term of $-L^{j_{k'}}(x^{k'})$ is $(-1)^j x_0 \cdots x_{j-1} a_{j-1} a_j \cdots a_{n-1}$. The j -face of this term is $x_0 \cdots x_{j-1} a_{j-1} \cdots a_{n-1}$, which cancels the j -face of $(-1)^{j+1} x_0 \cdots x_{j-1} x'_j a_j \cdots a_{n-1}$. Since the term of ω whose j -face cancels that of $-r_w x_0 \cdots x_{j-1} x_j a_j \cdots a_{n-1}$ is the term $r_w x_0 \cdots x_{j-1} x'_j a_j \cdots a_{n-1}$, we may assume without loss of generality that this case does not occur, i.e. that $j_{k'} = j$.

For ease of notation, denote $x^{k'}$ by x' ; then we have shown that ω contains terms of the form $-r_w L^j(x)$ and $r_w L^j(x')$. Moreover, by comparing the j -terms of the sums $L^j(x)$ and $L^j(x')$ we see that for $0 \leq i \leq j - 1$ we have $x_i = x'_i$, and for $j \leq i \leq n - 1$ we have $\pi x_i = \pi x'_i = a_i$, and hence

$$r_w \sum_{i=0}^{j-1} (-1)^i x_0 \cdots x_i a_i \cdots a_{n-1} = r_w \sum_{i=0}^{j-1} (-1)^i x'_0 \cdots x'_i a_i \cdots a_{n-1}.$$

We may therefore add their difference to ω without changing its value. Regrouping our terms, we obtain:

$$\begin{aligned} & \left(\sum_{i=0}^{j-1} (-1)^i r_w x'_0 \cdots x'_i a_i \cdots a_{n-1} + L^j(x') \right) - \left(\sum_{i=0}^{j-1} (-1)^i r_w x_0 \cdots x_i a_i \cdots a_{n-1} + L^j(x) \right) \\ &= r_w L^0(x') - r_w L^0(x). \end{aligned}$$

Repeating this procedure for all w , we obtain an expression for ω as a sum of q with a set of L -terms, in which fewer than m of the lower limits of the L -terms are nonzero. By induction, it follows that ω can be expressed in such a form with all lower limits equal to zero. \square

We are finally equipped to prove the following:

PROPOSITION 4.28. *The map $E: M(X, A) \rightarrow \widehat{\Omega}(X, A)$ is an isomorphism of chain complexes.*

Proof. We first show that each map $E: M_n(X, A) \rightarrow \widehat{\Omega}_n(X, A)$ is injective. To see this, let $(p, q) \in \widehat{\Omega}_{n-1}^1(X, A) \oplus \widehat{\Omega}(X, A)$ such that $q + L^0(p) = 0$. Rearranging this expression, we obtain $q = -L^0(p)$. Since every nonzero term of $L^0(p)$ contains a vertex of A while no nonzero term of q contains a vertex of A , it follows that both q and $L^0(p)$ are zero. Thus $p = 0$ by Lemma 4.16.

We now prove surjectivity of E . For some $n \geq 0$, let $\omega \in \widehat{\Omega}(X, A)$. By Lemma 4.27,

$$\omega = L^0(p) + q$$

for some $p \in \widehat{A}_{n-1}^1(X, A)$ and $q \in A_n(X - A)$. To prove that ω is in the image of E we must show that $p \in \widehat{\Omega}_{n-1}^1(X, A)$ and $q \in \Omega_n(X - A)$, i.e. that the boundaries of both p and q are allowed, and that all terms of the boundary of p have their last vertices in A .

We now proceed analogously to the proof of [14, Lem. 3.10]. By Lemma 4.22

$$\partial\omega = -L^0(\partial p) - p + \pi(p) + \partial q.$$

Rearranging this equation, we obtain:

$$\partial\omega + p - \pi(p) = -L^0(\partial p) + \partial q.$$

On the left-hand side of the equation, the terms $\partial\omega$ and p are allowed by assumption, while $\pi(p)$ is allowed by Lemma 4.21. Thus we see that $-L^0(\partial p) + \partial q$ is allowed. Now we may note that every nonzero term of $-L^0(\partial p)$ includes at least one vertex of A , while this is not the case for any term of ∂q . Thus there can be no cancellations between that terms of $-L^0(\partial p)$ and ∂q , implying that $L^0(p)$ and ∂q must each be allowed. Thus $q \in \Omega_n(X - A)$. Furthermore, by Lemma 4.19, the fact that $L^0(\partial p)$ is allowed implies $\partial p \in \widehat{A}_{n-2}^1(X, A)$. \square

COROLLARY 4.29. *The functors $Q, \widehat{\Omega}: \text{DiGraph}_2 \rightarrow \text{Ch}_R$ send all morphisms of $\text{DiGraph}_2^{\text{PO}}$ to isomorphisms.*

Proof. This follows immediately from Proposition 4.11, Lemma 4.14, and Proposition 4.28. \square

We can now prove the main result of this section.

Proof of Theorem 4.5. The relative homology maps $H_n(X, A) \rightarrow H_n(X', A')$ are induced by the map $\Omega(X)/\Omega(A) \rightarrow \Omega(X')/\Omega(A')$, i.e. the image of $(X, A) \rightarrow (X', A')$ under the functor $Q: \text{DiGraph}_2 \rightarrow \text{Ch}_R$. This is an isomorphism of chain complexes by Corollary 4.29. \square

5. MAIN THEOREM

In this section, we prove the main theorem of the paper:

THEOREM 5.1. *For any ring R , the category DiGraph of directed graphs admits the structure of a cofibration category, with the cofibrations as defined by Definition 3.9 and R -homology isomorphisms of Definition 2.21 as the weak equivalences.*

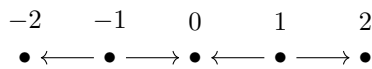
Most of the axioms of Definition 2.33 have already been proven in Sections 3 and 4; those which remain to be proven are the factorization of codiagonal maps and the closure of acyclic cofibrations under transfinite composition. We first consider the factorization axiom. As a preliminary, we note the following:

PROPOSITION 5.2. *A box product of homology isomorphisms is a homology isomorphism.*

Proof. This is immediate from [18, Thm. 4.7]. □

PROPOSITION 5.3. *For every $X \in \text{DiGraph}$, the codiagonal map $X \sqcup X \rightarrow X$ factors as a cofibration followed by a path homology isomorphism.*

Proof. Let J denote the directed graph pictured below:



Let ∂J denote the subcomplex of J consisting of the two endpoint vertices -2 and 2 . The inclusion $\partial J \hookrightarrow J$ is a cofibration, with projecting decomposition given by $\pi(-1) = \pi(-2) = -2, \pi(1) = \pi(2) = 2$. Furthermore, J has trivial path homology by Lemma 2.23.

Now observe that for any directed graph X , the codiagonal map $X \sqcup X \rightarrow X$ is isomorphic to the box product of the identity on X with the unique map $\partial J \rightarrow I_0$. Thus the codiagonal may be factored as:

$$X \sqcup \partial J \twoheadrightarrow X \sqcup J \rightarrow X$$

The map $X \sqcup \partial J \twoheadrightarrow X \sqcup J$ is the box product of the identity on X with the cofibration $\partial J \hookrightarrow J$, hence a cofibration by Proposition 3.14. Similarly, the map $X \sqcup J \rightarrow X$ is the box product of the identity on X with the homology isomorphism $J \rightarrow I_0$, hence a homology isomorphism by Proposition 5.2. □

Next we consider the stability of acyclic cofibrations under transfinite composition.

PROPOSITION 5.4. *For any ring R , the functor $\Omega: \text{DiGraph} \rightarrow \text{Ch}_R$ preserves filtered colimits.*

Proof. As colimits of chain complexes are computed component-wise, it suffices to check $\Omega_n: \text{DiGraph} \rightarrow \text{Mod}_R$ preserves filtered colimits for each $n \in \mathbb{N}$.

For a graph X , the R -module $\Omega_n X$ is the pullback:

$$\begin{array}{ccc} \Omega_n X & \longrightarrow & A_{n-1}(X) \\ \downarrow & \lrcorner & \downarrow \\ A_n(X) & \longrightarrow & C_{n-1}(X). \end{array}$$

As filtered colimits commute with finite limits, it suffices to show the functors

$$\begin{aligned} A_{n-1}(-): \text{DiGraph} &\rightarrow \text{Mod}_R \\ A_n(-): \text{DiGraph} &\rightarrow \text{Mod}_R \\ C_{n-1}(-): \text{DiGraph} &\rightarrow \text{Mod}_R \end{aligned}$$

preserve filtered colimits. This follows since the sets $\{0, \dots, n-1\}, \{0, \dots, n\}$ are finite and the graph I_{n-1} is finite. □

COROLLARY 5.5. *For any ring R , the path homology functor $H_*: \text{DiGraph} \rightarrow \text{Mod}_R^{\mathbb{N}}$ preserves filtered colimits.*

Proof. This is immediate from Proposition 5.4, together with the fact that the homology functor on chain complexes $H_*: \text{Ch}_R \rightarrow \text{Mod}_R^{\mathbb{N}}$ preserves filtered colimits. \square

PROPOSITION 5.6. *A transfinite composite of weak equivalences is a weak equivalence.*

Proof. By definition, weak equivalences are maps which become isomorphisms under the path homology functor $H_*: \text{DiGraph} \rightarrow \text{Mod}_R^{\mathbb{N}}$. The result then follows from Corollary 5.5. \square

Proof of Theorem 5.1. We consider each of the axioms of Definition 2.33.

- (C1) The class of cofibrations contains all identity maps and is closed under composition by Proposition 3.12. The analogous results for weak equivalences are immediate from the functoriality of path homology.
- (C2) The 2-out-of-6 property for weak equivalences is immediate from the corresponding property for isomorphisms and the functoriality of path homology.
- (C3) All objects of DiGraph are cofibrant by Proposition 3.13.
- (C4) The existence of pushouts of cofibrations is trivial, as DiGraph is cocomplete. Stability of cofibrations under pushout is given by Proposition 3.15. Given a pushout square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with $A \twoheadrightarrow X$ an acyclic cofibration, we can view it as a morphism in DiGraph_2 . It follows by Proposition 3.15 that $B \twoheadrightarrow Y$ is a cofibration as well. By Corollary 4.4, each relative homology group $H_n(X, A)$ is trivial; and by Theorem 4.5, so is each relative homology group $H_n(Y, B)$. Thus $B \twoheadrightarrow Y$ is a weak equivalence by Corollary 4.4.

- (C5) Factorization of codiagonal maps is given by Proposition 5.3.
- (C6) The existence of small coproducts is trivial, as DiGraph is cocomplete.
- (C7) The closure of (acyclic) cofibrations under transfinite composition follows from Proposition 3.18 and Proposition 5.6. \square

Our results also allow us to compare our cofibration category structure on DiGraph with the cofibration categories of chain complexes defined in Example 2.38.

THEOREM 5.7. *For any ring R , the functor $\Omega: \text{DiGraph} \rightarrow \text{Ch}_R$ factors through the full subcategory of chain complexes of projective R -modules, and is exact when considered as a functor from the cofibration category of Theorem 5.1 to either $\text{Ch}_R^{\text{proj}}$ or Ch_R .*

Proof. To see that Ω factors through $\text{Ch}_R^{\text{proj}}$, we note that for any directed graph X , each abelian group $\Omega_n(X)$ is free, as a subgroup of the free abelian group $C_n(X)$.

Now we consider exactness of Ω . It suffices to prove the statement for $\text{Ch}_R^{\text{proj}}$; the statement for Ch_R^{inj} will then follow by Example 2.41.

That Ω preserves weak equivalences is immediate, as the path homology isomorphisms of directed graphs are by definition the maps which Ω sends to quasi-isomorphisms. To see that Ω preserves the initial object, we observe that $\Omega(\emptyset)$ is zero in each degree.

For Ω to preserve cofibrations means that it sends cofibrations of directed graphs to inclusions of chain complexes with degreewise projective cokernel. This follows from Proposition 4.11 and the fact that for any cofibration $A \twoheadrightarrow X$, the abelian group $\widehat{\Omega}_n(X, A)$ is free as a subgroup of the free abelian group $C_n(X)$. That Ω preserves transfinite composites of cofibrations is immediate from Proposition 5.4.

To see that Ω preserves pushouts of cofibrations, consider a pushout square $f: (X, A) \rightarrow (X', A')$. To show that Ω sends this diagram to a pushout, it suffices to show that each functor Ω_n for $n \geq 0$ sends it to a pushout of abelian groups. By Lemma 4.7 and Corollary 4.29, the image of this diagram under Ω_n is isomorphic to:

$$\begin{array}{ccc} \Omega_n(A) & \longrightarrow & \Omega_n(A') \\ \downarrow & & \downarrow \\ \Omega_n(A) \oplus \widehat{\Omega}_n(X) & \longrightarrow & \Omega_n(A') \oplus \widehat{\Omega}_n(X) \end{array}$$

(Note that this is isomorphic to, yet distinct from, the diagram appearing in the proof of Lemma 4.7: in the first component of the bottom map we have replaced the isomorphism $\widehat{\Omega}_n(X, A) \cong \widehat{\Omega}_n(X', A')$ with the identity on $\widehat{\Omega}_n(X, A)$.)

Now consider the following composite diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \Omega_n(G) & \longrightarrow & \Omega_n(G') \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_n^U(H) & \longrightarrow & \Omega_n(G) \oplus \Omega_n^U(H) & \longrightarrow & \Omega_n(G') \oplus \Omega_n^U(H) \end{array}$$

The left square and the composite rectangle are pushouts, as the direct sum is the coproduct in the category of abelian groups. It follows that the right square is a pushout by the two pushout lemma. \square

It is natural to ask about the compatibility of our cofibrations with other classes of equivalences; for instance, we might ask whether they comprise part of a category structure whose weak equivalences are the homotopy equivalences of directed graphs. We next show that this is not the case. Preceding the proof, we recall the notions of homotopy and homotopy equivalence as in [16].

A *line digraph* of size n is any graph I whose vertex set is $0, 1, \dots, n$ and such that for any $i = 0, 1, \dots, n - 1$, we have either $i \rightarrow i + 1$ or $i + 1 \rightarrow i$. All digraphs I_n of Definition 2.6 are examples of line digraphs, but line digraphs also include the graph J used in the proof of Proposition 5.3 and, for example,

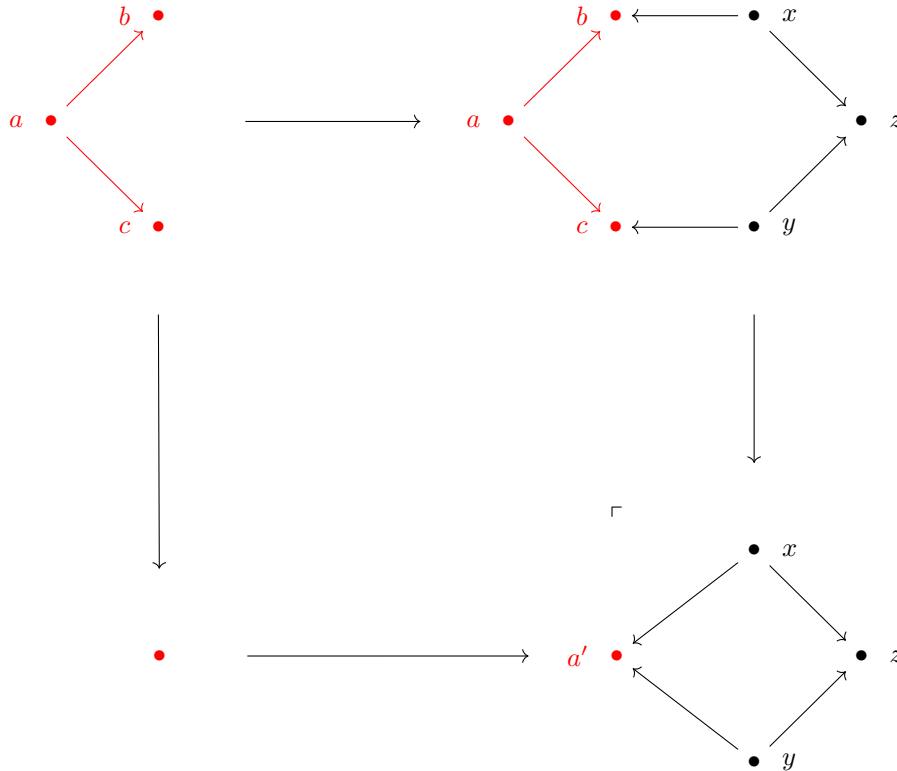
$$\begin{array}{ccccc} 0 & & 1 & & 2 \\ \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet \end{array}$$

Given digraph maps $f, g: X \rightarrow Y$, a *homotopy* from f to g , denoted $\alpha: f \sim g$, is a digraph map $\alpha: X \square I \rightarrow Y$ for some line digraph I of size n such that $\alpha(-, 0) = f$ and $\alpha(-, n) = g$. A digraph map $f: X \rightarrow Y$ is a *homotopy equivalence* if there exists a map $g: Y \rightarrow X$ and homotopies $\alpha: gf \sim \text{id}_X$ and $\beta: fg \sim \text{id}_Y$.

By Lemma 2.23, all line digraphs are homotopy equivalent to the point, i.e. contractible. On the other hand, the cycle graphs of different length are not homotopy equivalent. Indeed, any map from a cycle of smaller size to a cycle of larger size is homotopic to a constant, while the identity map from a cycle to itself is not.

PROPOSITION 5.8. *There is no cofibration category structure on DiGraph in which the class of cofibrations includes the maps of Definition 3.9 and whose weak equivalences are the homotopy equivalences.*

Proof. Consider the following pushout of digraphs:



Note that the left vertical map is a homotopy equivalence and the top horizontal map is a cofibration. If the proposed cofibration category structure were to exist, the right vertical map would be a homotopy equivalence by left properness (Lemma 2.36). However, since it is a map between cycles of different sizes, it cannot be a homotopy equivalence. Hence, the proposed cofibration category structure does not exist. \square

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