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Triangulations of root polytopes

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Abstract Let Φ be an irreducible crystallographic root system and $\mathcal{P}$ its root polytope, i.e., the convex hull of Φ. We provide a uniform construction, for all root types, of a triangulation of the facets of $\mathcal{P}$. We also prove that, on each orbit of facets under the action of the Weyl group, the triangulation is unimodular with respect to a root sublattice that depends on the orbit.

1. Introduction

Let Φ be an irreducible crystallographic root system in a Euclidean space E, Φ^+ a positive system of Φ, and W the Weyl group of Φ. Then, let $\mathcal{P}$ be the *root polytope* associated with Φ, i.e. the convex hull of all roots in Φ.

In [6], Marietti and the author have studied a natural set of representatives of the faces of $\mathcal{P}$ modulo the action of W, the *standard parabolic* faces of $\mathcal{P}$. The set of all roots contained in a standard parabolic face is an abelian ideal of Φ^+ (see Subsection 2.3 for a definition). We call *face ideals* or *facet ideals* the abelian ideals of Φ^+ corresponding to the standard parabolic faces or facets of $\mathcal{P}$.

In [4], for Φ of type A_n and C_n, the same authors have constructed a triangulation of the standard parabolic facets whose simplexes have a natural interpretation in terms of the corresponding facet ideals. The construction is formally equal for both root types, though the proofs are distinct and based on the special combinatorics of these two root systems and their maximal abelian ideals. Through the action of W, a triangulation of all the standard parabolic facets can be extended to a triangulation of the boundary of $\mathcal{P}$. Such an extension corresponds to an appropriate choice of representatives of the left cosets of W modulo the stabilizers of the standard parabolic facets. The triangulations of the boundary of $\mathcal{P}$ are also studied in [1] for all classical root types, using the coordinate description of Φ. In [12], the triangulations of the positive root polytope $\mathcal{P}^+$, i.e the convex hull of the positive roots and the origin, are studied for Φ of type A_n. The triangulations of $\mathcal{P}^+$ are also studied in [18, 19] for A_n and C_n.

In this paper, we give a uniform construction of a triangulation of the standard parabolic facets, for all finite irreducible crystallographic root systems. The construction coincides with the one of [4] for the types A_n and C_n. We also obtain unimodularity results similar to those obtained for A_n and C_n.

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We need some preliminaries for describing the results in more detail. If β_1, β_2, γ_1, $\gamma_2 \in \Phi^+$ are such that $\beta_1+\beta_2=\gamma_1+\gamma_2$, we say that $\{\beta_1,\beta_2\}$ and $\{\gamma_1,\gamma_2\}$ are *crossing pairs*. We first prove that if $\{\beta_1,\beta_2\},\{\gamma_1,\gamma_2\}$ are crossing pairs contained in a (common) abelian ideal, then, for all i,j in $\{1,2\}$, the differences $\beta_i-\gamma_j$ are roots, in particular β_i and γ_j are comparable. This implies that the set $\{\beta_1,\beta_2,\gamma_1,\gamma_2\}$ has a minimum and a maximum, more precisely, one of the two crossing pairs consists of these minimum and maximum, i.e., either $\beta_1<\gamma_i<\beta_2$ for both $i=1$ and 2, or the analogous relation with β and γ interchanged holds. We define the relations $\lesssim$ and $\sim$ on Φ^+ as follows. For all β_1,β_2 in Φ^+, we write $\beta_1\lesssim\beta_2$ if there exist γ_1,γ_2 such that $\beta_1+\beta_2=\gamma_1+\gamma_2$ and $\beta_1<\gamma_i<\beta_2$ for both $i=1$ and 2. Moreover, we write $\beta_1\sim\beta_2$ if $\beta_1\lesssim\beta_2$ or $\beta_2\lesssim\beta_1$. Finally, we say that a subset R of Φ^+ is *reduced* if $\beta_1\not\sim\beta_2$ for all $\beta_1,\beta_2\in R$.

The first main result in this paper is that the maximal reduced subsets in a facet ideal provide a triangulation of the corresponding standard parabolic facet. For each standard parabolic facet F of $\mathcal{P}$, let I_F be the corresponding facet ideal:

$$I_F = F\cap\Phi,$$

and

$$\mathcal{T}_F=\{\mathrm{Conv}(R)\mid R\subseteq I_F,\ R \text{ maximal reduced}\},$$

where $\mathrm{Conv}(R)$ is the convex hull of R. Then the following result holds.

Theorem 1.1. *For each standard parabolic facet F of $\mathcal{P}$, $\mathcal{T}_F$ is a triangulation of F.*

Clearly, the set of vertexes of the above triangulation is the set of all roots contained in F.

Theorem 1.1 implies, in particular, that the maximal reduced subsets in I_F are linear bases of E. Let Π and θ be the simple system and the highest root of Φ^+. Then, $\{-\theta\}\cup\Pi$ is the set of vertexes of the affine Dynkin diagram of Φ. For each $\alpha\in\Pi$, let Φ_α and $\widehat{\Phi}_\alpha$ be the root subsystems of Φ generated by $\Pi\smallsetminus\{\alpha\}$ and $\{-\theta\}\cup(\Pi\smallsetminus\{\alpha\})$, respectively, and Φ_α^+ and $\widehat{\Phi}_\alpha^+$ their positive systems contained in Φ^+. Clearly, $\widehat{\Phi}_\alpha$ has the same rank as Φ. We call the $\widehat{\Phi}_\alpha$, for all $\alpha\in\Pi$, the *standard equal rank* subsystems of Φ. The standard parabolic facets of $\mathcal{P}$ naturally correspond to the *irreducible* standard equal rank root subsystems of Φ [6]. Precisely, for each $\alpha\in\Pi$ such that $\widehat{\Phi}_\alpha$ is irreducible, let

$$I_\alpha=\widehat{\Phi}_\alpha^+\smallsetminus\Phi_\alpha.$$

Then I_α is a facet ideal of Φ^+, and each facet ideal of Φ^+ is obtained in this way (see Subsection 2.5). We prove the following result.

Theorem 1.2. *Let $\alpha\in\Pi$ be such that $\widehat{\Phi}_\alpha$ is irreducible. Then, each maximal reduced subset contained in the facet ideal I_α is a $\mathbb{Z}$-basis of the root lattice of $\widehat{\Phi}_\alpha$. In particular, all the simplexes of the triangulation $\mathcal{T}_F$ have the same volume.*

Part of the proofs require a case by case analysis. The cases to be considered can be restricted to a special, proper subset of facet ideals. Indeed, the results of [6] imply that the facet ideal I_α ($\alpha\in\Pi$, $\widehat{\Phi}_\alpha$ irreducible), is an *abelian nilradical* (see Subsection 2.4) in the root subsystem $\widehat{\Phi}_\alpha^+$. Hence, we may reduce to the case of abelian nilradicals.

The case by case analysis is contained in the proof of Proposition 5.11. This proof also provides an algorithm for the explicit computation of the triangulations for each root type, which will be done in a future paper.

2. Preliminaries

In this section we fix our main notation and recall some preliminary results. For the basic preliminary notions, we refer to [2] and [14] for root systems, and to [3] and [13] for Lie algebras.

2.1. Basic notation. *General.* We sometimes use the symbol $:=$ for emphasizing that equality holds by definition or that we are defining the left term of equality. We denote by E a Euclidean space, with scalar product $(\cdot\,,\cdot)$ and norm $|\cdot|$. We identify E with its dual space, through $(\cdot\,,\cdot)$. The null vector of E is denoted by $\underline{0}$. For any $S \subseteq \mathrm{E}$, $\mathrm{Span}(S)$ is the vector subspace generated by S over $\mathbb{R}$ (the field of real numbers), and $\mathrm{rk}(S) := \dim \mathrm{Span}(S)$.

Root systems. We denote by Φ a reduced irreducible crystallographic root system in E and by Φ^+ a fixed positive system of Φ. The simple system of Φ corresponding to Φ^+ is denoted by Π, while $\Omega^\vee$ is the set of fundamental co-weights of Φ, i.e., the dual basis of Π in E. For each $\alpha \in \Pi$, $\breve{\omega}_\alpha$ is the fundamental co-weight defined by the conditions $(\alpha, \breve{\omega}_\alpha) = 1$ and $(\alpha', \breve{\omega}_\alpha) = 0$ for all $\alpha' \in \Pi \setminus \{\alpha\}$. For all $\alpha \in \Pi$ and $\beta \in \Phi$, we set $c_\alpha(\beta) = (\beta, \breve{\omega}_\alpha)$, so that

$$\beta = \sum_{\alpha \in \Pi} c_\alpha(\beta)\alpha.$$

The *support* of β is the set of simple roots with non-zero coefficient in the expression of β:

$$\mathrm{Supp}(\beta) = \{\alpha \in \Pi \mid c_\alpha(\beta) \neq 0\}.$$

The highest root in Φ^+ is denoted by θ and its coefficients with respect to Π by m_α, thus

$$\theta = \sum_{\alpha \in \Pi} m_\alpha \alpha.$$

We call m_α the *multiplicity of α in Φ^+*.

For all $\beta \in \Phi$, $\beta^\vee$ is the corresponding coroot, i.e., $\beta^\vee = \frac{2\beta}{(\beta,\beta)}$.

For each root subsystem Ψ of Φ we set $\Psi^+ = \Psi \cap \Phi^+$. It is well known that Ψ^+ is a positive system for Ψ: we call it the *standard positive system* of Ψ. Moreover, we denote by $L(\Psi)$ and $L^+(\Psi)$ the root lattice and positive root lattice of Ψ, i.e. the $\mathbb{Z}$-span of Ψ and the $\mathbb{N}$-span of Ψ^+, respectively, where $\mathbb{Z}$ and $\mathbb{N}$ are the sets of integers and non-negative integers.

For any $S \subseteq \Phi$, we denote by $\Phi(S)$ the root subsystem of Φ generated by S, i.e., the minimal root system containing S, and we write $\Phi^+(S)$ for $\Phi(S)^+$.

A root subsystem Ψ of Φ is called *parabolic* if $\Psi = \Phi \cap \mathrm{Span}(\Psi)$. For any linear subspace H of E, the intersection $\Phi \cap H$ is a parabolic root subsystem. Hence, Ψ is a parabolic subsystem of Φ if and only if there exists a linear subspace H of E such that $\Psi = \Phi \cap H$.

Posets. As usual, $\leqslant$ denotes both the order of $\mathbb{R}$ and the partial order of E associated to Φ^+: for all $x, y \in \mathrm{E}$, $x \leqslant y$ if and only if $y - x \in L^+(\Phi)$. We call this last order *the standard partial order*. We will need only the restriction of the standard partial order to Φ^+. For any $S \subseteq \Phi^+$, we denote by $\mathrm{Min}\, S$ and $\mathrm{Max}\, S$, with capital M, the sets of minimal and maximal elements of S, and by $\min S$ and $\max S$ its possible minimum and maximum, with respect to $\leqslant$. The analogous objects with respect to any other order relation $\preccurlyeq$, will be distinguished by adding the subscript $\preccurlyeq$. The elements in $\mathrm{Min}\, S \cup \mathrm{Max}\, S$ are called *the extremal* elements of S. We say that S is *saturated* if it is saturated with respect to the standard partial order, i.e., for all $\beta_1, \beta_2 \in S$ such that $\beta_1 \leqslant \beta_2$, all the interval $[\beta_1, \beta_2] := \{\gamma \in \Phi \mid \beta_1 \leqslant \gamma \leqslant \beta_2\}$ is contained in S. Any subset S' of S is called *an initial section of S* if for all $\beta \in S'$ and $\gamma \in S$, if $\gamma \leqslant \beta$, then $\gamma \in S'$. The final sections are defined similarly. Then, S' is an initial section of S if and only if $S \smallsetminus S'$ is a final section of S.

For any order relation $\preccurlyeq$ on Φ^+ and for all $\beta \in \Phi^+$, we denote $(\beta^{\preccurlyeq})$ the $\preccurlyeq$*-upper cone of* β, i.e.,

$$(\beta^{\preccurlyeq}) = \{\gamma \in \Phi^+ \mid \beta \preccurlyeq \gamma\}.$$

Clearly, this is a *dual order ideal*, or *filter*, in the poset $(\Phi^+, \preccurlyeq)$.

2.2. Basic lemmas on roots.

2.2.1. *General facts.* We first recall some basic facts that we will use also without explicit mention. Since we are assuming Φ irreducible and reduced, the lengths of roots in Φ are at most 2 [2, Ch. VI, § 1.4]. We denote by Φ_ℓ the set of roots of maximal length (*long roots*), and set $\Phi_s = \Phi \smallsetminus \Phi_\ell$ (the set of *short roots*). By definition, if only one length occurs, all roots are long. Results (1) to (4) below can be found in [2, Ch. VI, § 1, n. 3, 4, 5].

(1) For all $\beta, \gamma \in \Phi$, if $(\beta, \gamma) < 0$ and $\beta \neq -\gamma$, then $\beta + \gamma \in \Phi$. Equivalently, if $(\beta, \gamma) > 0$ and $\beta \neq \gamma$, then $\beta - \gamma \in \Phi$.
(2) For all $\beta, \gamma \in \Phi$, if $\beta \neq \gamma$ and either $\gamma \in \Phi_\ell$, or $|\beta| = |\gamma|$, then $(\beta, \gamma^\vee) \in \{0, \pm 1\}$. If $|\beta| > |\gamma|$, then $\frac{(\beta,\beta)}{(\gamma,\gamma)} \in \{2, 3\}$ and $(\beta, \gamma^\vee) = \frac{(\beta,\beta)}{(\gamma,\gamma)}(\beta^\vee, \gamma) \in \{0, \pm 2, \pm 3\}$.
(3) For all $\beta, \gamma \in \Phi$, the set $I = \{k \in \mathbb{Z} \mid \beta + k\gamma \in \Phi\}$ is a an interval of $\mathbb{Z}$.
(4) For all $\alpha, \alpha' \in \Pi$, $(\alpha, \alpha') \leqslant 0$.

We note that in (2), since the root lengths are at most two, we have either $\frac{(\beta,\beta)}{(\gamma,\gamma)} = 2$ and $(\beta, \gamma^\vee) \in \{0, \pm 2\}$ for all $\beta, \gamma \in \Phi$ such that $|\beta| > |\gamma|$, or $\frac{(\beta,\beta)}{(\gamma,\gamma)} = 3$ and $(\beta, \gamma^\vee) \in \{0, \pm 3\}$ for all $\beta, \gamma \in \Phi$ such that $|\beta| > |\gamma|$.

2.2.2. *Summable roots.* Next proposition contains a less known result. The proof requires some basic notions and results from Lie theory. Let $\mathfrak{g}$ be a complex simple Lie algebra with root system Φ with respect to the Cartan subalgebra $\mathfrak{h}$ (see e.g. [13, §18]). Thus, $\mathfrak{g} = \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h}$, where $\mathfrak{g}_\alpha$ is the root space of α, for all $\alpha \in \Phi$, and $(\mathrm{Span}_{\mathbb{C}}(\Phi)) = \mathfrak{h}^*$, the dual space of $\mathfrak{h}$.

We say that two roots are *summable* if their sum is a root. It is well known that if α and β are summable roots, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$, while if α and β are not summable and $\alpha \neq -\beta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{\underline{0}\}$.

Proposition 2.1. *Let $\beta_1, \beta_2, \beta_3 \in \Phi$ be such that $\beta_1 + \beta_2 + \beta_3 \in \Phi$ and $\beta_i \neq -\beta_j$ for all $i, j \in \{1, 2, 3\}$. Then at least two of the three sums $\beta_i + \beta_j$, with $i, j \in \{1, 2, 3\}$ and $i \neq j$, belong to Φ.*

Proof. For all $i \in \{1, 2, 3\}$, we have $\beta_1 + \beta_2 + \beta_3 \neq \beta_i$, otherwise, for $\{j, k\} = \{1, 2, 3\} \smallsetminus \{i\}$, we have $\beta_j + \beta_k = \underline{0}$, contrary to the assumption. Moreover, for at least one $i \in \{1, 2, 3\}$ we have $(\beta_1 + \beta_2 + \beta_3, \beta_i) > 0$, since $(\beta_1 + \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_3) > 0$, hence $\beta_1 + \beta_2 + \beta_3 - \beta_i \in \Phi$. Assume for example $\beta_1 + \beta_2 \in \Phi$. Then, $[[\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_2}], \mathfrak{g}_{\beta_3}] = [\mathfrak{g}_{\beta_1+\beta_2}, \mathfrak{g}_{\beta_3}] = \mathfrak{g}_{\beta_1+\beta_2+\beta_3} \neq \{\underline{0}\}$, hence, by the Jacobi identity, at least one of $[[\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_3}], \mathfrak{g}_{\beta_2}]$ and $[\mathfrak{g}_{\beta_1}, [\mathfrak{g}_{\beta_2}, \mathfrak{g}_{\beta_3}]]$ is non zero. It follows that at least one of $\beta_1 + \beta_3$ and $\beta_2 + \beta_3$ is a root. $\square$

As we have recalled, if two roots have strictly negative scalar product, then they are summable. The reverse implication holds if Φ is simply laced, but is false in general, as we see in the following lemma, where the Cartan integers of pairs of summable roots are classified.

Lemma 2.2. *Assume $\beta, \gamma, \beta+\gamma \in \Phi$.*

(1) *If $|\beta| = |\gamma| = |\beta+\gamma|$, then $(\beta, \gamma^\vee) = -1$.*
(2) *If $|\beta| = |\gamma| \neq |\beta+\gamma|$, then either $\frac{|\beta+\gamma|^2}{|\beta|^2} = 2$ and $(\beta, \gamma^\vee) = 0$, or $\frac{|\beta+\gamma|^2}{|\beta|^2} = 3$ and $(\beta, \gamma^\vee) = 1$. In any case, $|\beta| = |\gamma| < |\beta+\gamma|$.*
(3) *If $|\beta| < |\gamma|$, then $|\beta+\gamma| = |\beta|$, $(\beta^\vee, \gamma) = -\frac{|\gamma|^2}{|\beta|^2} \in \{-2, -3\}$, and $(\beta, \gamma^\vee) = -1$.*

Proof. (1) and (2). Since $|\beta| = |\gamma|$, we have $\frac{|\beta+\gamma|^2}{|\gamma|^2} = \frac{(\beta,\beta)+2(\beta,\gamma)+(\gamma,\gamma)}{(\gamma,\gamma)} = 2 + \frac{2(\beta,\gamma)}{(\gamma,\gamma)} = 2 + (\beta, \gamma^\vee)$. Moreover, in case (1) we have $\frac{|\beta+\gamma|^2}{|\gamma|^2} = 1$, while in case (2) we have $\frac{|\beta+\gamma|^2}{|\gamma|^2} = \frac{|\beta+\gamma|^2}{|\beta|^2} \in \{2, 3\}$. In all cases, the claim follows directly.

(3) If $|\beta+\gamma| = |\gamma|$, we get $1 = \frac{|\beta+\gamma|^2}{|\gamma|^2} = \frac{(\beta,\beta)}{(\gamma,\gamma)} + (\beta, \gamma^\vee) + 1$, hence $\frac{(\beta,\beta)}{(\gamma,\gamma)} = -(\beta, \gamma^\vee) \in \mathbb{Z}$, contrary to the assumption. Hence, $|\beta+\gamma| = |\beta|$, and $1 = \frac{|\beta+\gamma|^2}{|\beta|^2} = 1 + (\beta^\vee, \gamma) + \frac{|\gamma|^2}{|\beta|^2}$. Hence, $(\beta^\vee, \gamma) = -\frac{|\gamma|^2}{|\beta|^2} \in \{-2, -3\}$. Finally, since $(\beta, \gamma^\vee) = \frac{|\beta|^2}{|\gamma|^2}(\beta^\vee, \gamma)$, we have $(\beta, \gamma^\vee) = -1$ □

The assumptions of parts (1), (2), and (3) of Lemma 2.2 are mutually exclusive and cover all possibilities for the relations among $|\beta|$, $|\gamma|$, and $|\beta+\gamma|$. In particular, if β and γ have nonnegative scalar product, then, by parts (1) and (3), we obtain that the assumptions of part (2) holds. Similarly, if γ is long, then the assumptions of parts (1) or (3) hold. Hence, we obtain the following proposition.

Proposition 2.3. *For all $\beta, \gamma \in \Phi$, the following results holds.*

(1) *If β and γ are summable, then we have $(\beta, \gamma) \geqslant 0$ if and only if $|\beta| = |\gamma| < |\beta+\gamma|$.*
(2) *If $\gamma \in \Phi_\ell$, then β and γ are summable if and only if $(\beta, \gamma^\vee) = -1$.*

2.3. Ad-nilpotent and abelian ideals. Let $\mathfrak{g}$ be as in Subsection 2.2, $\mathfrak{b}$ be the standard Borel subalgebra of $\mathfrak{g}$ associated to Φ^+, and $\mathfrak{n}$ its nilpotent radical, i.e., $\mathfrak{b} = \left(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h}$ and $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$.

An *ad-nilpotent* ideal of $\mathfrak{b}$ is a (nilpotent) ideal of $\mathfrak{b}$ contained in $\mathfrak{n}$. Being $\mathfrak{h}$-stable, such an ideal is a sum of root spaces. For any $I \subseteq \Phi^+$, the sum of root spaces $\bigoplus_{\alpha\in I} \mathfrak{g}_\alpha$ is an ad-nilpotent ideal of $\mathfrak{b}$ if and only if, for all $\alpha, \beta \in \Phi^+$, if $\alpha \in I$ and $\alpha \leqslant \beta$, then $\beta \in I$. A subset I of Φ^+ with this property is called an *ad-nilpotent ideal* of Φ^+. Thus, an ad-nilpotent ideal of Φ^+ is a filter in $(\Phi^+, \leqslant)$, i.e. a dual order ideal. It is easy to see that an abelian ideal of $\mathfrak{b}$ must be ad-nilpotent. For any $I \subseteq \Phi^+$, the subspace $\bigoplus_{\alpha\in I} \mathfrak{g}_\alpha$ is an abelian ideal of $\mathfrak{b}$ if and only if I is an ad-nilpotent ideal of Φ^+ with the further property that, for all $\alpha, \beta \in I$, $\alpha + \beta \notin \Phi$. Such an I is called an *abelian ideal* of Φ^+. The abelian ideals of Φ^+ are studied in several papers, both for their implications in representation theory and for their algebraic-combinatorial interest. The main representation theoretic motivations can be found in [16, 17] (see also [7]); the basic algebraic-combinatorial results can be found in [8, 9, 20, 21].

2.4. Abelian nilradicals. An ad-nilpotent ideal of Φ^+ is called *principal* if it has a minimum, i.e. if the corresponding $\mathfrak{b}$-ideal is principal. For all $\beta \in \Phi^+$, the upper $\leqslant$-cone of β, $(\beta^{\leqslant}) = \{\gamma \in \Phi^+ \mid \beta \leqslant \gamma\}$, is also called the principal ad-nilpotent ideal *generated* by β. It is clear that if $\beta \in \Phi^+$ is such that $c_\alpha(\beta) > \frac{m_\alpha}{2}$ for some $\alpha \in \Pi$, then $(\beta^{\leqslant})$ is abelian. In particular, this happens if β is a simple root of multiplicity 1 in Φ^+. Indeed, the following well known result holds. For completeness, we include a proof.

Proposition 2.4. *Let $S \subseteq \Pi$ and $I = \Phi^+ \setminus \Phi(S)$. Then I is an ad-nilpotent ideal. Moreover, I is abelian if and only if either $S = \Pi$, or $S = \Pi \setminus \{\alpha\}$ for a simple root α such that $m_\alpha = 1$. In this case, I is equal to $(\alpha^{\leqslant})$ and is a maximal abelian ideal.*

Proof. It is immediate that I is an ad-nilpotent ideal. If $S = \Pi$, then I is the empty root ideal, hence it is abelian. Let $S = \Pi \setminus \{\alpha\}$ with $\alpha \in \Pi$ and $m_\alpha = 1$. Then, by definition we have $I = (\alpha^{\leqslant})$, which is abelian since $m_\alpha = 1$. We prove that $(\alpha^{\leqslant})$ is maximal abelian. If $S = \varnothing$, i.e. $\Pi = \{\alpha\}$, then $I = \Phi^+$ and the claim is obvious. Let $S \neq \varnothing$ and let J be an ad-nilpotent ideal that strictly contains $(\alpha^{\leqslant})$. We have to prove that J is not abelian. By definition, there exists $\beta \in J$ such that $\alpha \notin \mathrm{Supp}(\beta)$. Let $\Psi_1, \dots, \Psi_k$ be the irreducible components of $\Phi(\Pi \setminus \{\alpha\})$. Assume, for example, $\beta \in \Psi_1$. Then, if θ_1 is the highest root of Ψ_1, we obtain $\theta_1 \in J$. Let $S_1 = \Psi_1 \cap (\Pi \setminus \{\alpha\})$. It is easily seen that, since Φ is irreducible, α cannot be orthogonal to the whole S_1. Hence, since $(\alpha, \alpha') \leqslant 0$ for all $\alpha' \in S_1$, there exists $\alpha' \in S_1$ such that $(\alpha', \alpha) < 0$. But $\mathrm{Supp}(\theta_1) = S_1$, hence $(\theta_1, \alpha) \leqslant (\alpha', \alpha) < 0$. It follows that $\theta_1 + \alpha \in \Phi$, and hence J is not abelian.

It remains to prove the "only if" part. For all $\beta \in \Phi$, let $\mathrm{ht}_{\Pi \setminus S}(\beta) = \sum\limits_{\alpha \in \Pi \setminus S} c_\alpha(\beta)$. We have $S = \Pi$ if and only if $\max\{\mathrm{ht}_{\Pi \setminus S}(\beta) \mid \beta \in \Phi\} = 0$. Similarly, we have $S = \Pi \setminus \{\alpha\}$ and $m_\alpha = 1$ if and only if $\max\{\mathrm{ht}_{\Pi \setminus S}(\beta) \mid \beta \in \Phi\} = 1$. In order to conclude the proof, we assume $\max\{\mathrm{ht}_{\Pi \setminus S}(\beta) \mid \beta \in \Phi\} > 1$ and prove that in this case I is not abelian. By definition, we have $\beta \in I$ if and only if $\mathrm{ht}_{\Pi \setminus S}(\beta) > 0$. Let $\beta^* \in \mathrm{Min}\{\beta \in \Phi \mid \mathrm{ht}_{\Pi \setminus S}(\beta) > 1\}$. Since $(\beta^*, \beta^*) > 0$, there exists $\alpha \in \mathrm{Supp}(\beta^*)$ such that $(\beta^*, \alpha) > 0$, hence $\beta^* - \alpha \in \Phi$, by 2.2.1(1). Such an α cannot belong to S, otherwise $\mathrm{ht}_{\Pi \setminus S}(\beta^* - \alpha) = \mathrm{ht}_{\Pi \setminus S}(\beta^*)$, contrary to minimality of β^*. It follows $\alpha \in \Pi \setminus S$, hence $\alpha \in I$. Now, $\beta^* - \alpha \in I$, too, since $\mathrm{ht}_{\Pi \setminus S}(\beta^* - \alpha) = \mathrm{ht}(\beta^*) - 1 > 0$, hence we obtain that I is not abelian since α and $\beta^* - \alpha$ are summable. □

For each $S \subseteq \Pi$, the ideal $\bigoplus\limits_{\alpha \in \Phi^+ \setminus \Phi(S)} \mathfrak{g}_\alpha$ is the nilradical (the largest nilpotent ideal) of the standard parabolic subalgebra associated to S (see [3, Ch. VIII, § 3.4]). Hence, we call the maximal abelian ideals $(\alpha^{\leqslant})$ with $m_\alpha = 1$, together with the empty root ideal, *the abelian nilradicals*.

2.5. The faces of the root polytope. We recall some ideas and results from [6]. For all $\alpha \in \Pi$ and all nonempty $S \subseteq \Pi$, let

$$H_{\alpha, m_\alpha} = \{x \in E \mid (x, \check{\omega}_\alpha) = m_\alpha\}, \qquad \mathrm{F}_\alpha = H_{\alpha, m_\alpha} \cap \mathcal{P}, \qquad \mathrm{F}_S = \bigcap_{\alpha \in S} \mathrm{F}_\alpha.$$

By definition of m_α, we have $(\beta, \check{\omega}_\alpha) \leqslant m_\alpha$ for all $\beta \in \Phi$ and, therefore, $(x, \check{\omega}_\alpha) \leqslant m_\alpha$ for all $x \in \mathcal{P}$. Hence, the affine hyperplanes H_{α, m_α} are supporting hyperplanes of $\mathcal{P}$, and the F_α and F_S are faces of $\mathcal{P}$. We call them the *standard parabolic faces*. In fact, the set of all standard parabolic faces is a set of representatives of the orbits of the action of the Weyl group W on the set of proper faces of $\mathcal{P}$ [6] .

For each standard parabolic face F, let

$$I_F = F \cap \Phi.$$

By definition, for each nonempty $S \subseteq \Pi$, I_{F_S} is the set of all roots β such that $c_\alpha(\beta) = m_\alpha$, for all $\alpha \in S$. It is easy to see that $\mathcal{P}$ is the convex hull of the long roots (see e.g. [5]), hence the long roots in I_{F_S} are the vertexes of the face F_S.

We recall that the *extended* Dynkin graph of Φ is obtained from the usual Dynkin graph by extending the vertex set Π with $-\theta$ and completing the edge set according to the scalar products and the relative lengths between $-\theta$ and the roots in Π, with the same rules used of the usual Dynkin graph. For our purposes, it is convenient to consider the extended Dynkin graph on the opposite vertex set, i.e., $\{\theta\} \cup -\Pi$. This does not change the edges. We call the resulting graph the *opposite* extended Dynkin graph.

For each $\Sigma \subseteq \Pi$, we set

$$\Sigma^{\mathrm{e}} = \{\theta\} \cup -\Sigma.$$

Let $\Phi(\Sigma^{\mathrm{e}})$ be the root subsystem of Φ generated by Σ^{e}. For studying the face F_S, we need considering $\Phi(\Sigma^{\mathrm{e}})$ with $\Sigma = \Pi \setminus S$. In this case, $\Sigma \subsetneqq \Pi$, since S is assumed to be nonempty. (We point out that, contrary to what is done in [6], we are not making use of the affine root system associated to Φ and all root subsystems we are defining are inside Φ.)

It is well known that, if $\Sigma \subsetneqq \Pi$, then Σ^{e} is a simple system for $\Phi(\Sigma^{\mathrm{e}})$ [11, Ch. II, § 5].

In general, $\Phi(\Sigma^{\mathrm{e}})$ is not irreducible. Let $\Sigma^{\mathrm{e}}_\theta$ be the subset of Σ^{e} defined by the condition that $\Phi(\Sigma^{\mathrm{e}}_\theta)$ is the irreducible component of $\Phi(\Sigma^{\mathrm{e}})$ that contains θ. Finally, let $\Sigma_\theta = \Sigma^{\mathrm{e}}_\theta \setminus \{\theta\}$. We denote by $\Gamma(\Sigma^{\mathrm{e}})$ and $\Gamma(\Sigma^{\mathrm{e}}_\theta)$ the subgraphs induced by Σ^{e} and $\Sigma^{\mathrm{e}}_\theta$ in the opposite Dynkin graph oh Φ. Then for $\Sigma \subsetneqq \Pi$, we have that $\Gamma(\Sigma^{\mathrm{e}})$ is the Dynkin graph of $\Phi(\Sigma^{\mathrm{e}})$, and $\Gamma(\Sigma^{\mathrm{e}}_\theta)$ is the connected component of θ in $\Gamma(\Sigma^{\mathrm{e}})$.

The following proposition contains the preliminary results on the standard parabolic faces that we need. We note that the proposition also precises that the face F_S does not determine S. In fact, by parts (1) or (2), for all $S, S' \subseteq \Pi$, we have $\mathrm{F}_S = \mathrm{F}_{S'}$ if and only if $\Phi^+((\Pi \setminus S)^{\mathrm{e}}_\theta) = \Phi^+((\Pi \setminus S')^{\mathrm{e}}_\theta)$, i.e., F_S is uniquely determined by the irreducible component $\Phi^+((\Pi \setminus S)^{\mathrm{e}}_\theta)$. In particular, the standard parabolic faces, and therefore the W-orbits of faces, are in bijection with the proper connected subgraphs of the opposite extended Dynkin graph that contains the vertex θ [22].

Proposition 2.5 ([6]). *Let $S \subseteq \Pi$, $S \neq \varnothing$.*

(1) $I_{\mathrm{F}_S} = \Phi^+((\Pi \setminus S)^{\mathrm{e}}) \setminus \Phi(\Pi \setminus S) = \Phi^+((\Pi \setminus S)^{\mathrm{e}}_\theta) \setminus \Phi((\Pi \setminus S)_\theta)$.

(2) *Let μ_S be the highest root of $\Phi((\Pi \setminus S)^{\mathrm{e}}_\theta)$, with respect to the simple system $(\Pi \setminus S)^{\mathrm{e}}_\theta$. Then, I_{F_S} is the principal abelian ideal of Φ^+ generated by μ_S.*

(3) $\dim(\mathrm{F}_S) = |(\Pi \setminus S)_\theta|$.

By definition of I_{F_S}, part (2) says that μ_S is the unique minimal root such that $c_\alpha(\mu_S) = m_\alpha$ for all $\alpha \in S$. Both (1) and (2) implies that we have $c_\alpha(\mu_S) < m_\alpha$ if and only if $\alpha \in (\Pi \setminus S)_\theta$. Hence, for all $\beta \in \Phi^+$, the condition $c_\alpha(\beta) = m_\alpha$ for all $\alpha \in S$ implies $c_\alpha(\beta) = m_\alpha$ also for all $\alpha \in \Pi \setminus (\Pi \setminus S)_\theta$, which in general is greater than S.

Definition 2.6. *We call the ideals I_{F_S}, for all nonempty $S \subseteq \Pi$, face ideals. The face ideals corresponding to the facets are also called facet ideals.*

Definition 2.7. *We denote by $\widetilde{\Phi^+}((\Pi \setminus S)^{\mathrm{e}})$ the positive system of $\Phi((\Pi \setminus S)^{\mathrm{e}})$ relative to the simple system $(\Pi \setminus S)^{\mathrm{e}}$. Similarly, we denote by $\widetilde{\Phi^+}((\Pi \setminus S)^{\mathrm{e}}_\theta)$ the positive system of $\Phi((\Pi \setminus S)^{\mathrm{e}}_\theta)$ relative to the simple system $(\Pi \setminus S)^{\mathrm{e}}_\theta$.*

Remark 2.8. For each $S \neq \varnothing$, the positive system $\widetilde{\Phi^+}((\Pi \setminus S)^{\mathrm{e}})$ is different from $\Phi^+((\Pi \setminus S)^{\mathrm{e}})$, which is the intersection $\Phi(\Pi \setminus S) \cap \Phi^+$, by definition. However, we have $\Phi^+((\Pi \setminus S)^{\mathrm{e}}) \setminus \Phi(\Pi \setminus S) = \widetilde{\Phi^+}((\Pi \setminus S)^{\mathrm{e}}) \setminus \Phi(\Pi \setminus S)$. The same considerations hold for $(\Pi \setminus S)_\theta$ in place of $\Pi \setminus S$. Therefore, in Proposition 2.5(1) we may replace Φ^+ with $\widetilde{\Phi^+}$.

By the above remark, Proposition 2.5(1) is equivalent to the following corollary.

Corollary 2.9. *The set I_{F_S} is the principal ideal generated by θ in the positive system $\widetilde{\Phi^+}((\Pi \setminus S)^{\mathrm{e}}_\theta)$ of the irreducible root system $\Phi((\Pi \setminus S)^{\mathrm{e}}_\theta)$.*

2.6. The order involution of face ideals. For all $w \in W$, let

$$N(w) = \{\beta \in \Phi^+ \mid w(\beta) \leqslant \underline{0}\}.$$

For all $\Sigma \subseteq \Pi$, let $w_{0,\Sigma}$ be the longest element in the standard parabolic subgroup of W generated by $\{s_\alpha \mid \alpha \in \Sigma\}$. It is well known that $w_{0,\Sigma}$ is an involution and is determined by the condition $N(w_{0,\Sigma}) = \Phi^+(\Sigma)$.

Proposition 2.10. *Let $\varnothing \neq S \subseteq \Pi$ and $w_S^* = w_{0,(\Pi\smallsetminus S)}$. Then, the restriction of w_S^* to I_{F_S} is an anti-isomorphism of the poset $(I_{\mathrm{F}_S}, \leqslant)$. In particular, w_S^* exchange θ and μ_S.*

Proof. We observe that, by definition, $I_{\mathrm{F}_S} = (\theta + L(\Phi(\Pi\smallsetminus S))) \cap \Phi$. For all $\alpha \in \Pi \smallsetminus S$, we have $s_\alpha(\theta) \in \theta + L(\Phi(\Pi \smallsetminus S))$, hence we easily obtain $s_\alpha(I_{\mathrm{F}_S}) = I_{\mathrm{F}_S}$. It follows $w_S^*(I_{\mathrm{F}_S}) = I_{\mathrm{F}_S}$.

It remains to prove that w_S^* reverses the standard partial order on I_{F_S}. Let $\beta, \beta' \in I_{\mathrm{F}_S}$ and $\beta < \beta'$. Then $\beta' - \beta \in L^+(\Phi(\Pi\smallsetminus S))$, and since $w_S^*(\alpha) < \underline{0}$ for all $\alpha \in (\Pi\smallsetminus S)$, $w_S^*(\beta') - w_S^*(\beta) = w_S^*(\beta - \beta') \in -L^+(\Phi(\Pi\smallsetminus S))$, i.e. $w_S^*(\beta') < w_S^*(\beta)$, as claimed. □

We note that, by Proposition 2.5(1), the above proposition holds also with $w_{0,(\Pi\smallsetminus S)_\theta}$ in place of w_S^*. In particular, the restrictions of $w_{0,(\Pi\smallsetminus S)_\theta}$ and of w_S^* on I_{F_S} coincide.

Definition 2.11. *We call w_S^* the face involution of F_S and the restriction of w_S^* to I_{F_S} the order involution of I_{F_S}.*

3. Face ideals and abelian nilradicals

In this section we prove that the abelian nilradicals of Φ^+ are facet ideals and that all face ideals are abelian nilradicals in some irreducible subsystem of Φ.

By Proposition 2.5, the standard parabolic facets of $\mathcal{P}$ are the faces of type F_α with $\alpha \in \Pi$ such that $\Phi((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$ is irreducible. Equivalently, F_α is a facet if and only if α does not disconnect the extended Dynkin diagram, when removed. For type A_n, $\Phi((\Pi\smallsetminus\{\alpha\})^{\mathrm{e}})$ is irreducible for all $\alpha \in \Pi$. For all other root types, $\Phi((\Pi\smallsetminus\{\alpha\})^{\mathrm{e}})$ is irreducible if and only if α is a leaf of the extended Dynkin diagram.

In the next proposition we see that if $m_\alpha = 1$, then $\Phi((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$ is irreducible, hence I_{F_α} is a facet ideal. We note that in this case $I_{F_\alpha} = (\alpha^{\leqslant})$.

Proposition 3.1. *Each nonempty abelian nilradical of Φ^+ is a facet ideal.*

Proof. It is well known that if α is any simple root such that $m_\alpha = 1$, then the subgraph of the extended Dynkin graph obtained by removing α is isomorphic to the (ordinary) Dynkin graph of Φ [15]. In particular, $\Phi((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$ is irreducible, hence $(\Pi \smallsetminus \{\alpha\})^{\mathrm{e}}_\theta = (\Pi \smallsetminus \{\alpha\})^{\mathrm{e}}$ and $(\Pi \smallsetminus \{\alpha\})_\theta = \Pi \smallsetminus \{\alpha\}$. By Proposition 2.5(3), we have $\dim(\mathrm{F}_\alpha) = |\Pi \smallsetminus \{\alpha\}| = n-1$, i.e., F_α is a facet. □

If $m_\alpha = 1$, then α is the minimum of I_{F_α}, hence, the order involution $w_{0,\Pi\smallsetminus\{\alpha\}}$ maps α onto θ. Since it also maps $\Pi \smallsetminus \{\alpha\}$ onto $-(\Pi \smallsetminus \{\alpha\})$, it maps Π onto the nodes of the opposite extended Dynkin graph minus the node $-\alpha$. Hence, the fact that $\Phi((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$ is isomorphic to Φ for all α with $m_\alpha = 1$, is a consequence of Proposition 2.10.

By a direct check, we can see that, for the root types A_n, C_n, D_n, and E_6, we have that $\Phi((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$ is irreducible if and only if $m_\alpha = 1$. For the other root types, there exists at least a leaf $\alpha \in \Pi$ of the extended Dynkin diagram such that $m_\alpha > 1$. Then, F_α is a facet, but I_{F_α} is not an abelian nilradical in Φ. Thus, the converse of Proposition 3.1 is not true. However, the following result holds.

Proposition 3.2. *Each face ideal in Φ^+ is an abelian nilradical of some irreducible root subsystem of Φ.*

Proof. By Corollary 2.9, any face ideal I_{F_S} ($\varnothing \neq S \subseteq \Pi$) is the principal ideal generated by θ in the positive system $\widetilde{\Phi^+}((\Pi \smallsetminus S)^{\mathrm{e}}_\theta)$.

By Definition 2.7, the simple system of $\widetilde{\Phi^+}((\Pi \smallsetminus S)^{\mathrm{e}}_\theta)$ is $(\Pi \smallsetminus S)^{\mathrm{e}}_\theta$, and $(\Pi \smallsetminus S)^{\mathrm{e}}_\theta = \{\theta\} \cup (\Pi \smallsetminus S)_\theta$, where $(\Pi \smallsetminus S)_\theta$ is a certain subset of $-(\Pi \smallsetminus S)$. Hence, for all $\beta \in \widetilde{\Phi^+}((\Pi \smallsetminus S)^{\mathrm{e}})_\theta$, we have $\beta = c_\theta \theta - \sum\limits_{\alpha \in \Pi \smallsetminus S} c_\alpha \alpha$, for some nonnegative c_θ, c_α. This implies that, for all $\alpha \in S$, $c_\alpha(\beta) = c_\theta m_\alpha$ and, hence, $c_\theta \leqslant 1$. In other words, the multiplicity of θ, as a simple root in the positive system $\widetilde{\Phi^+}((\Pi \smallsetminus S)^{\mathrm{e}})$, is 1. Hence, the principal ideal generated by θ in $\widetilde{\Phi^+}((\Pi \smallsetminus S)^{\mathrm{e}})$ is an abelian nilradical. $\square$

Remark 3.3. Let $\alpha \in \Pi$ be such that F_α is a facet. By Proposition 2.5, I_{F_α} is also equal to $(\mu_{\{\alpha\}}^{\leqslant})$, where $\mu_{\{\alpha\}}$ is the unique root in Φ such that $c_\alpha(\mu_{\{\alpha\}}) = m_\alpha$ and $c_{\alpha'}(\mu_{\{\alpha\}}) < m_{\alpha'}$ for all $\alpha' \in \Pi \smallsetminus \{\alpha\}$. By Proposition 2.10, the face involution $w^*_{\{\alpha\}}$ maps $(\Pi \smallsetminus \{\alpha\})^{\mathrm{e}}$ onto $\{\mu_{\{\alpha\}}\} \cup (\Pi \smallsetminus \{\alpha\})$, therefore, this last set is a simple system for $\Phi((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$. In fact, $\{\mu_{\{\alpha\}}\} \cup (\Pi \smallsetminus \{\alpha\})$ is the simple system of the positive system $\Phi^+((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$.

Since $w^*_{\{\alpha\}}(\theta) = \mu_{\{\alpha\}}$, by the proof of Proposition 3.2 we obtain that the multiplicity of $\mu_{\{\alpha\}}$, as a simple root in $\Phi^+((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$, is 1. Hence, I_{F_α} is the abelian nilradical generated by $\mu_{\{\alpha\}}$ in the positive system $\Phi^+((\Pi \smallsetminus \{\alpha\})^{\mathrm{e}})$.

The definition of ad-nilpotent and abelian ideals makes sense also in the reducible case. Let Ψ be any finite crystallographic root system, $\Psi_1, \dots, \Psi_k$ be its irreducible components, Ψ_i^+ a positive system for Ψ_i, for $i = 1, \dots, k$, and $\Psi^+ = \Psi_1^+ \cup \dots \cup \Psi_k^+$. Then, by definition, I is an ad-nilpotent, or abelian, ideal of Ψ^+ if and only if $I \cap \Psi_i^+$ is an ad-nilpotent, or abelian, ideal of Ψ_i^+ for all $i \in \{1, \dots, k\}$. Moreover, I is an abelian nilradical of Ψ^+ if and only if $I \cap \Psi_i^+$ is an abelian nilradical of Ψ_i^+ for all $i \in \{1, \dots, k\}$. This means that $I \cap \Psi_i^+$ is either empty or a principal ideal generated by a simple root with multiplicity 1.

Proposition 3.4. *Let I be an abelian nilradical of Φ^+ and Ψ a root subsystem of Φ. Then $I \cap \Psi$ is an abelian nilradical of Ψ^+.*

Proof. Let $\Psi_1 \dots, \Psi_k$, for $i = 1, \dots, k$, be the irreducible components of Ψ. We have to prove that $I \cap \Psi_i$ is an abelian nilradical of Ψ_i^+ for $i = 1, \dots, k$, hence we may directly assume that Ψ is irreducible.

Let $I = (\alpha^{\leqslant})$, with $\alpha \in \Pi$ and $m_\alpha = 1$. Let Π_Ψ be the simple system of Ψ^+, and $\theta_\Psi = \sum_{\beta \in \Pi_\Psi} m'_\beta \beta$ be the highest root in Ψ.

If $I \cap \Psi = \varnothing$ we are done.

If there exists some β in $I \cap \Psi$, then $\beta \leqslant \theta_\Psi$ and hence $\theta_\Psi \in I$, i.e., $c_\alpha(\theta_\Psi) = 1$. Let $\Pi_\Psi(\alpha) = I \cap \Pi_\Psi$. By definition, we have $c_\alpha(\beta) = 1$ for all $\beta \in \Pi_\Psi(\alpha)$. Moreover, since $\Pi_\Psi \subseteq \Phi^+$, we have $c_\alpha(\beta) = 0$ for all $\beta \in \Pi_\Psi \smallsetminus I$. Hence, $1 = c_\alpha(\theta_\Psi) = \sum_{\beta \in \Pi_\Psi(\alpha)} m'_\beta$. Since all coefficients m'_β are strictly positive, we obtain that there exists a unique root β^* such that $\Pi_\Psi(\alpha) = \{\beta^*\}$. Moreover, we have $m'_{\beta^*} = 1$, i.e., by definition, β^* has multiplicity 1 as a simple root of Ψ. Finally, for all $\gamma \in \Psi$, let $\gamma = \sum_{\beta \in \Pi_\Psi} c'_\beta(\gamma)\beta$ be the decomposition of γ with respect to the basis Π_Ψ. Then, $c_\alpha(\gamma) = c'_{\beta^*}(\gamma)$, hence, $\gamma \in I \cap \Psi$ if and only if $c'_{\beta^*}(\gamma) = 1$. It follows that $I \cap \Psi$ is the principal ideal generated by β^* in Ψ^+, and hence it is an abelian nilradical of Ψ^+. $\square$

From the above proof, we obtain also the following refinement of Proposition 3.4.

Proposition 3.5. *Let I be an abelian nilradical of Φ^+, Ψ an irreducible root subsystem of Φ, and Π_Ψ be the simple system of Ψ^+. Then:*

(1) *$I \cap \Psi \neq \varnothing$ if and only if $I \cap \Pi_\Psi \neq \varnothing$;*
(2) *if $I \cap \Pi_\Psi \neq \varnothing$, then $I \cap \Pi_\Psi$ consists of a single element;*
(3) *if $I \cap \Pi_\Psi = \{\beta^*\}$, then β^*, as a simple root of Ψ^+, has multiplicity 1, and $I \cap \Psi$ is the abelian nilradical generated by β^* in Ψ^+, i.e. $I \cap \Psi = \Psi^+ \setminus \Phi(\Pi_\Psi \setminus \{\beta^*\})$.*

4. Crossing pairs

In this section we analyze the properties of crossing pairs contained in abelian ideals. In the simply laced case, many of the results that we are proving could be proved in a very simpler way.

Definition 4.1. *Let $\beta_i, \gamma_i \in \Phi$, $i = 1, 2$, with $\beta_i \neq \gamma_j$ for all $i, j \in \{1, 2\}$. We say that $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are crossing pairs if $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$. In this case we call the equality $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$ a crossing relation. We do not assume that $\beta_1 \neq \beta_2$ and $\gamma_1 \neq \gamma_2$, hence (at most) one of the pairs $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ may be a multiset of a single root with multiplicity 2.*

Lemma 4.2. *Let I be an abelian ideal in Φ^+, and $\beta, \gamma \in I$.*

(1) *If $\beta \in \Phi_s$, $x \in \Phi$, and $\beta + x \in I$, then $x \in \Phi_s$.*
(2) *If $\beta - \gamma \in \Phi$, then $(\beta, \gamma) > 0$.*

Proof. (1) We prove that $|\beta| \geqslant |x|$, which yields the claim. By contradiction, let $|\beta| < |x|$. Then, by Lemma 2.2(3), $(x, \beta^\vee) \in \{-2, -3\}$. It follows $s_\beta(x) = x - (x, \beta^\vee)\beta \geqslant x + 2\beta$, hence $x + 2\beta \in \Phi$, which is contrary to the fact that I is abelian, since $x + 2\beta = \beta + (x + \beta)$ and $\beta, x + \beta \in I$.

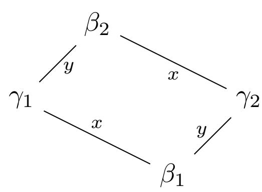
(2) By Proposition 2.3(1), applied to the summable roots β and $-\gamma$, we have $(\beta, -\gamma) \geqslant 0$ if and only if $\beta, \gamma \in \Phi_s$ and $\beta - \gamma \in \Phi_\ell$. By part (1) this cannot happen. Indeed, since $\gamma + (\beta - \gamma) \in I$, if $\gamma \in \Phi_s$, we must have $\beta - \gamma \in \Phi_s$. Therefore, $(\beta, -\gamma) < 0$, which gives the claim. ☐

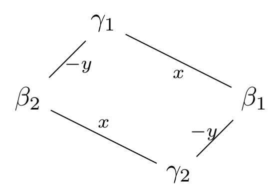
Proposition 4.3. *Let I be an abelian ideal in Φ^+ and $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$ be crossing pairs contained in I such that $\beta_1 \neq \beta_2$. Then:*

(1) *for all $i, j \in \{1, 2\}$ we have $(\beta_i, \gamma_j) > 0$, in particular $\beta_i - \gamma_j$ is a root;*
(2) *either $\{\beta_1, \beta_2\}$, or $\{\gamma_1, \gamma_2\}$ is the pair of the minimum and maximum of $\{\beta_i, \gamma_i \mid i = 1, 2\}$;*
(3) *$(\beta_1, \beta_2) = 0$ unless both of β_1, β_2 are short and γ_1, γ_2 have different lengths.*

Proof. (1) If $\{i, i'\} = \{1, 2\}$, we have $\beta_1 + \beta_2 - \gamma_i = \gamma_{i'} \in \Phi$. Moreover, since I is abelian, $\beta_1 + \beta_2 \notin \Phi$. By Proposition 2.1, applied to the summable triad $\beta_1, \beta_2, -\gamma_i$, we obtain $\beta_j - \gamma_i \in \Phi$ for $j \in \{1, 2\}$. By Lemma 4.2(2), it follows $(\beta_j, \gamma_i) > 0$ for $i, j \in \{1, 2\}$.

(2) We set $x = \gamma_1 - \beta_1 = \beta_2 - \gamma_2$ and $y = \gamma_2 - \beta_1 = \beta_2 - \gamma_1$. By part (1), x and y are roots. If x and y are both positive or both negative, we directly obtain that $\{\beta_1, \beta_2\}$ is the set of the minimum and maximum of $\{\beta_i, \gamma_i \mid i = 1, 2\}$. Similarly, if one of x, y is positive and the other is negative, $\{\gamma_1, \gamma_2\}$ is the set of the minimum and maximum $\{\beta_i, \gamma_1 \mid i = 1, 2\}$. (In the picture below we illustrate the Hasse diagram of the quadruple $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$ in the cases $x, y > \underline{0}$ and $x > \underline{0}, y < \underline{0}$.)





(3) We keep the notation of part (2). First, we assume that at least one of β_1, β_2, is long and prove that then $(\beta_1, \beta_2) = 0$. Let β_1 be long. Then, by Proposition 2.3 (2), applied to the two pairs of summable roots $\beta_1, -\gamma_2$ and β_1, x, we have $-(\beta_1^\vee, \gamma_2) = (\beta_1^\vee, x) = -1$. Hence, $(\beta_1^\vee, \beta_2) = (\beta_1^\vee, \gamma_2 + x) = 0$, which yields the claim. The case β_2 long is similar.

Now, we assume $\beta_1, \beta_2 \in \Phi_s$ and $(\beta_1, \beta_2) \neq 0$, and we prove that $|\gamma_1| \neq |\gamma_2|$. Since I is abelian, β_1 and β_2 are not summable, hence cannot have negative scalar product. Therefore $(\beta_1, \beta_2) > 0$, and since $|\beta_1| = |\beta_2|$, by 2.2.1(2) we have $(\beta_1^\vee, \beta_2) = 1$. By definition, we have $\beta_2 = \gamma_1 + y = \beta_1 + x + y$, hence $1 = (\beta_1^\vee, \beta_2) = (\beta_1^\vee, \beta_1) + (\beta_1^\vee, x) + (\beta_1^\vee, y) = 2 + (\beta_1^\vee, x) + (\beta_1^\vee, y)$. It follows $(\beta_1^\vee, x) + (\beta_1^\vee, y) = -1$. But, by Lemma 4.2(1), x and y are short, hence $(\beta_1^\vee, x), (\beta_1^\vee, y) \in \{0, \pm 1\}$ (by 2.2.1(2), again). Therefore $\{(\beta_1^\vee, x), (\beta_1^\vee, y)\} = \{0, -1\}$. We may assume $(\beta_1^\vee, x) = 0$ and $(\beta_1^\vee, y) = -1$, without loss of generality. Then, by Proposition 2.3(1), applied to the two summable pairs of short roots β_1, x and β_1, y, we obtain $|\beta_1| = |x| < |\beta_1 + x| = |\gamma_1|$, and $|\beta_1| = |y| \geqslant |\beta_1 + y| = |\gamma_2|$. Hence, γ_1 is long and γ_2 is short. ☐

Notation 4.4. We write $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$ for $\beta_1 < \gamma_i < \beta_2$ for both $i \in \{1, 2\}$.

Let $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ be crossing pairs. Up to exchange β_1 and β_2, we may assume $\beta_2 \not< \beta_1$. Similarly, without loss of generality, we may assume $\gamma_2 \not< \gamma_1$. Then, by Proposition 4.3(2), either $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, or $\gamma_1 < \{\beta_1, \beta_2\} < \gamma_2$.

Definition 4.5. *We define the relations $\lesssim$ and $\sim$ on Φ^+ as follows:*

$\beta_1 \lesssim \beta_2$ if and only if there exists $\gamma_1, \gamma_2 \in \Phi^+$ such that $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are crossing pairs with $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$;

$\beta_1 \sim \beta_2$ if and only if either $\beta_1 \lesssim \beta_2$ or $\beta_2 \lesssim \beta_1$.

If $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ are crossing pairs with $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, we also say that $\{\gamma_1, \gamma_2\}$ is a middle pair between β_1 and β_2 and that $\{\beta_1, \beta_2\}$ is a raising pair through γ_1 and γ_2.

In the next corollary we study the order relations among different raising pairs through a common middle pair and different middle pairs between a common raising pair.

Corollary 4.6. *Let I be an abelian ideal, $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ be crossing pairs in I with $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$.*

1. *If $\{\beta_1', \beta_2'\}$ is any other raising pair through $\{\gamma_1, \gamma_2\}$, with $\beta_1' < \beta_2'$, then either $\beta_1 < \beta_1' < \beta_2' < \beta_2$, or $\beta_1' < \beta_1 < \beta_2 < \beta_2'$. Moreover, $\beta_i - \beta_i' \in \Phi$ for both $i = 1, 2$.*
2. *If $\{\gamma_1', \gamma_2'\}$ is any other middle pair between $\{\beta_1, \beta_2\}$, then $\gamma_i - \gamma_j' \in \Phi$ for all $i, j \in \{1, 2\}$. Moreover, one of the following four cases occur: $\gamma_i' < \{\gamma_1, \gamma_2\} < \gamma_j'$, $\gamma_i < \{\gamma_1', \gamma_2'\} < \gamma_j$ (with $\{i, j\} = \{1, 2\}$). In particular, there exists at most one incomparable middle pair between β_1 and β_2.*

Proof. Under the assumption of (1), we have $\beta_1' + \beta_2' = \gamma_1 + \gamma_2 = \beta_1 + \beta_2$, hence $\{\beta_1', \beta_2'\}$ and $\{\beta_1, \beta_2\}$ are crossing pairs. Similarly, under the assumption of (2), $\{\gamma_1', \gamma_2'\}$ and

$\{\gamma_1, \gamma_2\}$ are crossing pairs. Hence, the claim follows directly from Proposition 4.3(2). $\square$

In the next lemma, we see that the possible lengths of roots and root differences in a crossing pair are very limited.

Lemma 4.7. *Let I be an abelian ideal in Φ^+, $\{\beta_1, \beta_2\}$, $\{\gamma_1, \gamma_2\}$ be crossing pairs contained in I, $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, $x = \beta_2 - \gamma_2 = \gamma_1 - \beta_1$, and $y = \beta_2 - \gamma_1 = \gamma_2 - \beta_1$.*

(1) *If either one of x, y is long, then x, y, β_1, β_2, γ_1, γ_2 are all long.*
(2) *If any one of x, y, β_1, β_2, γ_1, γ_2 is short, then x and y are short and at most one of β_1, β_2, γ_1, γ_2 is long, except when $\gamma_1 = \gamma_2$, in which case γ_1 is short and β_1, β_2 are long.*

Proof. We first prove that:

(a) if any one of β_1, β_2, γ_1, γ_2 is short, then x and y are short.

We provide the details for the case $\gamma_2 \in \Phi_s$. The other cases are similar. We have $\gamma_2 + x = \beta_2$, in particular $\gamma_2 + x \in I$, hence by Lemma 4.2(1), we obtain $x \in \Phi_s$. Similarly, since $\gamma_2 + (-y) = \beta_1 \in I$ we obtain $-y \in \Phi_s$, hence the claim.

Now we prove that:

(b) x and y are either both short, or both long.

It suffices to prove that if either one of x, y is short, then the other one is short, too. Assume, for example, $x \in \Phi_s$. By (a), it suffices to prove that at least one among β_i, γ_i ($i = 1, 2$), is short. If $\beta_1 \in \Phi_s$, we are done. Then, let $\beta_1 \in \Phi_\ell$. By Lemma 2.2(3), the sum of two roots of different lengths is always short, hence $\gamma_1 = \beta_1 + x$ is short and we are done. The case $y \in \Phi_s$ is similar, hence (b) is proved.

Now we conclude the proof of part (1). If either one of x, y is long, then, by (a), β_i and γ_i are long, for $i = 1, 2$. Moreover, by (b), both of x and y are long.

It remains to conclude the proof of part (2). So, we assume $\{x, y, \beta_1, \beta_2, \gamma_1, \gamma_2\} \not\subseteq \Phi_\ell$. Then, by part (1), $x, y \in \Phi_s$. We distinguish the two cases $\gamma_1 \neq \gamma_2$ and $\gamma_1 = \gamma_2$.

Let $\gamma_1 \neq \gamma_2$. We have to prove that if any root in $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$ is long, then the three remaining roots are short.

Let $\beta_1 \in \Phi_\ell$. By Lemma 2.2(3) the sum of a long and a short root is short, hence we obtain $\gamma_1, \gamma_2 \in \Phi_s$, since $\gamma_1 = \beta_1 + x$ and $\gamma_2 = \beta_1 + y$. Then, we have $\gamma_1, -x \in \Phi_s$, while $\gamma_1 + (-x) = \beta_1 \in \Phi_\ell$. By Proposition 2.3(1), this implies $(\gamma_1, -x) \geqslant 0$.

If $\beta_2 \in \Phi_\ell$, arguing in a similar way, we obtain $\gamma_1, \gamma_2 \in \Phi_s$. Moreover, $(\gamma_1, y) \geqslant 0$.

Now, if both $\beta_1, \beta_2 \in \Phi_\ell$, we deduce $(\gamma_1^\vee, \gamma_2) = (\gamma_1^\vee, \gamma_1 - x + y) \geqslant (\gamma_1^\vee, \gamma_1) = 2$. Since $|\gamma_1| = |\gamma_2|$, by 2.2.1(2) this implies $\gamma_1 = \gamma_2$, contrary to the assumption.

By a similar argument, taking into account that $\beta_1 \neq \beta_2$, we obtain that if one of γ_1, γ_2 is long, the three remaining roots in the crossing pairs are short, as claimed.

Now, let $\gamma_1 = \gamma_2$. We have to prove that $\gamma_1 \in \Phi_s$ and $\{\beta_1, \beta_2\} \subseteq \Phi_\ell$. Indeed, we have $\beta_1 + \beta_2 = 2\gamma_1$, hence $(\beta_1 + \beta_2, \gamma_1^\vee) = 4$. By 2.2.1(2), it follows that either $(\beta_1, \gamma_1^\vee) = (\beta_2, \gamma_1^\vee) = 2$, or $(\beta_1, \gamma_1^\vee) = 1$ and $(\beta_2, \gamma_1^\vee) = 3$. In the first case we obtain $\gamma_1 \in \Phi_s$ and $\beta_1, \beta_2 \in \Phi_\ell$, as claimed. The latter case cannot happen, otherwise we obtain $s_{\gamma_1}(-\beta_2) = -\beta_2 + 3\gamma_1 \in \Phi$, and $-\beta_2 + 3\gamma_1 = (-\beta_2 + 2\gamma_1) + \gamma_1 = \beta_1 + \gamma_1$, contrary to abelianity of I. $\square$

In the next proposition, we prove that, for any pair of comparable roots β_1 and β_2 in an abelian ideal I, if $\beta_1 - \beta_2$ is not a root, then $\beta_1 \sim \beta_2$. Moreover, we analyze when the reverse implication holds. We need the following well known result.

Lemma 4.8 ([2, Ch. VI, § 1.9, Proposition 19]). *Let $\gamma_1, \dots, \gamma_m \in \Phi^+$. If $\gamma_1 + \cdots + \gamma_m \in \Phi^+$, there exists a permutation $(\gamma_1', \dots, \gamma_m')$ of $(\gamma_1, \dots, \gamma_m)$ such that $\gamma_1' + \cdots + \gamma_h' \in \Phi$ for all $h \in \{1, \dots, m\}$.*

Proposition 4.9. *Let I be an abelian ideal in Φ^+ and $\beta_1, \beta_2 \in I$.*

(1) *If $\beta_1 < \beta_2$ and $\beta_2 - \beta_1 \notin \Phi$, then $\beta_1 \lesssim \beta_2$.*
(2) *If $\beta_1 \lesssim \beta_2$, $\{\beta_1, \beta_2\} \subseteq \Phi_s$ and there exists a middle pair $\{\gamma_1, \gamma_2\}$ between β_1, β_2 such that $\gamma_1 \in \Phi_s$ and $\gamma_2 \in \Phi_\ell$, then $\beta_2 - \beta_1 \in \Phi_s$.*
(3) *If $\beta_1 \lesssim \beta_2$, then $\beta_2 - \beta_1 \notin \Phi$ if and only if either one of the following conditions is satisfied:*
    (a) *at least one of β_1, β_2 is long,*
    (b) *$\{\beta_1, \beta_2\} \subseteq \Phi_s$ and there exists a middle pair $\{\gamma_1, \gamma_2\} \subseteq \Phi_s$ between β_1, β_2.*

Proof. (1) Let $\beta_1 < \beta_2$ and $\beta_2 - \beta_1 \notin \Phi$. By definition, $\beta_2 - \beta_1$ is a sum of positive roots. Let

$$k = \min\{h \in \mathbb{N} \mid \exists\, \eta_1, \dots, \eta_h \in \Phi^+ \text{ such that } \beta_2 - \beta_1 = \eta_1 + \cdots + \eta_h\},$$

and $\eta_1, \dots, \eta_k \in \Phi^+$ be such that $\beta_2 = \beta_1 + \eta_1 + \cdots + \eta_k$. By assumption, $k \geqslant 2$. Moreover, by minimality of k, no partial sum $\sum_{j=1}^{h} \eta_{i_j}$ with $1 \leqslant i_j \leqslant k$ and $h > 1$ is a root.

By Lemma 4.8, we may find a permutation $(\gamma'_1, \dots, \gamma'_m)$ of the sequence $(\beta_1, \eta_1, \dots, \eta_n)$ such that all partial sums $\gamma'_1 + \cdots + \gamma'_h$ are roots. By the above discussion, β_1 must be either γ'_1, or γ'_2. In both cases, we easily obtain that there exists a permutation $(\eta'_1, \dots, \eta'_k)$ of $(\eta_1, \dots, \eta_k)$ such that $\beta_1 + \sum\limits_{1 \leqslant j \leqslant h} \eta'_j \in \Phi$ for all $h \in \{0, \dots, k\}$.

Now, we prove that also $\beta_2 - \eta'_1 = \beta_1 + \sum\limits_{2 \leqslant j \leqslant k} \eta'_j \in \Phi$. This yields the crossing relation $\beta_1 + \beta_2 = (\beta_1 + \eta'_1) + (\beta_2 - \eta'_1)$, which concludes the proof of (1).

Let $\gamma_h = \beta_1 + \sum\limits_{1 \leqslant j \leqslant h} \eta'_j$, so that $\gamma_k = \beta_2$. We prove, by induction on h, that $\gamma_h - \eta_1 \in \Phi$ for all $h \in \{1, \dots, k\}$. For $h = 1$, the claim is clear, since $\gamma_1 = \beta_1 + \eta'_1$, by definition. Assume $1 \leqslant h < k$ and $\gamma_h - \eta'_1 \in \Phi$. We have

$$\gamma_{h+1} = \gamma_h + \eta'_{h+1} = (\gamma_h - \eta'_1) + \eta'_1 + \eta'_{h+1}.$$

By our assumption, $\eta'_1 + \eta'_{h+1} \notin \Phi$, hence, by Proposition 2.1, applied to the summable triad $(\gamma_h - \eta'_1), \eta'_1, \eta'_{h+1}$, we obtain that $(\gamma_h - \eta'_1) + \eta'_{h+1}$ is a root (and also $(\gamma_h - \eta'_1) + \eta'_1$ is a root). Since $(\gamma_h - \eta'_1) + \eta'_{h+1} = \gamma'_{h+1} - \eta'_1$, we get the claim.

(2) Let $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$, $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$, $\beta_1, \beta_2, \gamma_1 \in \Phi_s$, and $\gamma_2 \in \Phi_\ell$. We have to prove that $\beta_2 - \beta_1 \in \Phi_s$. First, we see that the condition $(\beta_2, \beta_1^\vee) > 0$ implies the claim. Indeed, by 2.2.1(1) if $(\beta_2, \beta_1^\vee) > 0$, then $\beta_2 - \beta_1 \in \Phi$. Moreover, $\beta_2 - \beta_1 \notin \Phi_\ell$, otherwise, by Proposition 2.3 (1), we should have $(\beta_2, -\beta_1^\vee) \geqslant 0$.

Thus, we prove that $(\beta_2, \beta_1^\vee) > 0$. Let $x = \beta_2 - \gamma_2 = \gamma_1 - \beta_1$. We recall that x is short, by Lemma 4.7 (1). Then, $(\beta_2, \beta_1^\vee) = (\gamma_2, \beta_1^\vee) + (x, \beta_1^\vee)$, and our assumptions on lengths imply $(x, \beta_1^\vee) \in \{0, \pm 1\}$ and $(\gamma_2, \beta_1^\vee) \in \{0, \pm 2, \pm 3\}$, by 2.2.1(2). Moreover, $(\gamma_2, \beta_1^\vee)$ is positive, by Proposition 4.3(1), hence, $(\gamma_2, \beta_1^\vee) \geqslant 2$ and $(\beta_2, \beta_1^\vee) \geqslant 2 - 1 = 1$, as claimed.

(3) For proving the "if" part, it suffices to prove that if neither (a), nor (b) hold, then the assumption of (2) holds. First, we assume that (a) does not hold, i.e., $\beta_1, \beta_2 \in \Phi_s$. Then, by Lemma 4.7 (2), for all middle pairs $\{\gamma_1, \gamma_2\}$ between β_1, β_2, at most one of γ_1, γ_2 is long. It follows that either (b) holds, or all middle pairs $\{\gamma_1, \gamma_2\}$ satisfies the assumption in (2).

It remains to prove the "only if" part. Let $\gamma_1, \gamma_2 \in \Phi$ be such that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$ and $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$.

(a) Assume $\beta_2 \in \Phi_\ell$ and, as before, let $x = \gamma_1 - \beta_1 = \beta_2 - \gamma_2$ and $y = \gamma_2 - \beta_1 = \beta_2 - \gamma_1$. Since the pairs of $\beta_2, -x$ and $\beta_2, -y$ are summable, by Proposition 2.3(2) we obtain $(x,\beta_2^\vee)=(y,\beta_2^\vee)=1$. Hence,

$$(*)\qquad 2=(\beta_2,\beta_2^\vee)=(\beta_1+x+y,\beta_2^\vee)=(\beta_1,\beta_2^\vee)+2.$$

It follows $(\beta_1,\beta_2^\vee)=0$ and, by Proposition 2.3(2), $\beta_2-\beta_1\notin\Phi$. If $\beta_1\in\Phi_\ell$, we may argue in a similar way and find again $\beta_2-\beta_1\notin\Phi$.

(b) Let $\beta_1,\beta_2\in\Phi_s$. By Proposition 4.7 (1), we have $x,y\in\Phi_s$, hence, by applying Lemma 2.2 (1) to the summable pairs $\beta_2,-x$ and $\beta_2,-y$, we obtain that equalities $(*)$ still hold. Hence, also in this case we have $\beta_2-\beta_1\notin\Phi$. □

The following corollary follows directly from parts (1) and (3) of Proposition 4.9.

Corollary 4.10. *Let I be an abelian ideal in Φ^+, and $\beta_1,\beta_2\in I$. If either one of β_1, β_2 is long, then $\beta_1\lesssim\beta_2$ if and only if $\beta_1<\beta_2$ and $\beta_2-\beta_1\notin\Phi$.*

If we combine parts (2) and (3) of Proposition 4.9, we obtain that the root lengths of all middle pairs between two fixed short roots β_1,β_2 are uniquely determined by β_1,β_2.

Corollary 4.11. *Let I be an abelian ideal in Φ^+, $\beta_1,\beta_2\in I\cap\Phi_s$, and $\beta_1\lesssim\beta_2$. Then either $\gamma_1,\gamma_2\in\Phi_s$ for all middle pairs $\{\gamma_1,\gamma_2\}$ between β_1,β_2, or γ_1 and γ_2 have different lengths for all middle pairs $\{\gamma_1,\gamma_2\}$ between β_1,β_2.*

Proof. Let $\{\gamma_1,\gamma_2\}$ and $\{\gamma_1',\gamma_2'\}$ be middle pairs between β_1,β_2. By Proposition 4.7(2), at least one of γ_1,γ_2 and at least one of γ_1',γ_2' are short. If $\{\gamma_1,\gamma_2\}\not\subseteq\Phi_s$ then, by part (2) of Proposition 4.9, we have $\beta_2-\beta_1\in\Phi_s$, while if $\{\gamma_1',\gamma_2'\}\subseteq\Phi_s$, then, by part (3) we have $\beta_2-\beta_1\notin\Phi$. Hence the two possibilities mutually exclude each other. □

Definition 4.12. *For any $S\subseteq\Phi^+$, we say that S is reduced if, for all $\beta,\beta'\in S$, $\beta\not\sim\beta'$.*

For all $\beta\in\Phi^+$ we set

$$\mathrm{Red}(\beta)=\{\beta'\in\Phi^+\mid\beta\neq\beta'\text{ and }\beta\not\sim\beta'\}.$$

Remark 4.13. By Proposition 4.9(1) and Lemma 4.2(2), if I is an abelian ideal, $\beta\in I$, and $I(\beta^{\lesseqgtr})=\{\gamma\in I\mid\gamma\text{ is comparable with }\beta\}$, then

$$(1)\qquad \mathrm{Red}(\beta)\cap I(\beta^{\lesseqgtr})\subseteq\{\gamma\in I\mid\gamma-\beta\in\Phi^+\}=\{\gamma\in I\setminus\{\beta\}\mid(\gamma,\beta)>0\}.$$

If $\beta\in\Phi_\ell$, in particular in the simply laced case, the inclusion is an equality, by Corollary 4.10. Moreover, if $\beta\in\Phi_\ell$, we have $(\gamma,\beta^\vee)\in\{0,\pm1\}$ for all $\gamma\in\Phi\setminus\{\beta\}$, hence

$$(2)\qquad \mathrm{Red}(\beta)\cap I(\beta^{\lesseqgtr})=\{\gamma\in I\mid(\gamma,\beta^\vee)=1\}.$$

In general, the inclusion in (1) is proper. As an example, for Φ of type C_n, if we number the simple roots as in [2] and take $I=(\alpha_n^{\leqslant})$, $\beta_1=\alpha_n+\alpha_{n-1},\beta_2=\alpha_n+2\alpha_{n-1}+\alpha_{n-2},\gamma_1=\alpha_n+\alpha_{n-1}+\alpha_{n-2},\gamma_2=\alpha_n+2\alpha_{n-1}$, we have: $\beta_1+\beta_2=\gamma_1+\gamma_2$, hence $\beta_1\lesssim\beta_2$, but $\beta_2-\beta_1=\alpha_{n-1}+\alpha_{n-2}\in\Phi_s$. This is an example of case (2) of Proposition 4.9.

5. Triangulation orders

In this section we define some special orderings of abelian ideals, which we call triangulation orders, and prove that all facet ideals have a triangulation order. Throughout the section, let I be an abelian ideal of Φ^+ such that $\mathrm{rk}(I)=n$. By “hyperplane”, we mean “linear hyperplane”.

Definition 5.1. *Let $J \subseteq I$. We say that J is bipartite if it has an initial section J_{i}, and a final section J_{f} such that*

(1) $J = J_{\mathrm{i}} \cup J_{\mathrm{f}}$*;*
(2) *for all $\beta_1 \in J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$ and $\beta_2 \in J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}$, we have $\beta_1 \lesssim \beta_2$;*
(3) *there exists a hyperplane H in E such that $J_{\mathrm{i}} \cap J_{\mathrm{f}} \subseteq H$ and H strictly separates $J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$ from $J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$.*

If the above conditions hold, we say that $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ is a bipartition of J. If, moreover, both J_{i} and J_{f} are proper subsets of J, we say that $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ is a proper bipartition. A hyperplane H as in (3) is called a separating hyperplane, for the bipartition $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ of J.

Note that, by definition, if J has a proper bipartition, then it has at least two elements. If J is also saturated, then it contains two crossing pairs and these provide at least three elements in J. The definition also implies that, if $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ is a bipartition of J, then $J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$ and $J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}$ are an initial and a final section of J, respectively, since the complement of an initial section is a final section, and vice-versa, and we have $J = (J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}) \sqcup J_{\mathrm{f}} = J_{\mathrm{i}} \sqcup (J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}})$, where by $\sqcup$ we denote disjoint union. Finally, we note that if J is saturated, also all the subsets J_{i}, J_{f}, $J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$, $J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}$, and $J_{\mathrm{i}} \cap J_{\mathrm{f}}$ are saturated.

Definition 5.2. *For each subset S of Φ^+, we define the restricted relations $\lesssim_S$ and $\sim_S$ on S as follows. For all $\beta_1, \beta_2 \in S$ we set:* (1) *$\beta_1 \lesssim_S \beta_2$ if there exists a middle pair $\{\gamma_1, \gamma_2\}$ between β_1 and β_2 contained in S;* (2) *$\beta_1 \sim_S \beta_2$ if either $\beta_1 \lesssim_S \beta_2$, or $\beta_2 \lesssim_S \beta_1$. We say that S is $\sim$closed if, for all $\beta_1, \beta_2 \in S \cap \Phi^+$, $\beta_1 \lesssim \beta_2$ implies $\beta_1 \lesssim_S \beta_2$.*

Obviously, for any $S \subseteq \Phi^+$ and $\beta_1, \beta_2 \in S$, the relation $\beta_1 \lesssim_S \beta_2$ implies $\beta_1 \lesssim \beta_2$. Hence, if S is $\sim$closed, then, for all $\beta_1, \beta_2 \in S$, we have $\beta_1 \sim \beta_2$ if and only if $\beta_1 \sim_S \beta_2$.

The first of following lemmas is clear, hence we omit the proof.

Lemma 5.3. *Let $S \subseteq \Phi^+$. If S is saturated, then S is $\sim$closed.*

Lemma 5.4. *Let I be an abelian ideal in Φ, Ψ a root subsystem of Φ, and $\Psi_1, \dots, \Psi_k$ be the irreducible components of Ψ. Moreover, let J be a $\sim$closed subset of Φ such that $J \subseteq I \cap \Psi$, and let $R \subseteq J$. Then, R is reduced in Φ if and only if $R \cap \Psi_i$ is reduced in Ψ_i for all $i \in \{i, \dots, k\}$.*

Proof. The "only if" part is obviuos. Conversely, we assume that $R \cap \Psi_i$ is reduced in Ψ_i for all $i \in \{i, \dots, k\}$ and prove that R is reduced in Φ. By contradiction, let $\beta_1 \lesssim \beta_2$ for some $\beta_1, \beta_2 \in R$. Then, since J is $\sim$closed, there exists a middle pair $\{\gamma_1, \gamma_2\}$, between β_1 and β_2, contained in J, hence in Ψ. By Proposition 4.3(1), $(\beta_j, \gamma_{j'}) > 0$ for all $j, j' \in \{1, 2\}$, hence, there exists $i \in \{1, \dots, k\}$ such that $\beta_j, \gamma_j \in \Psi_i$, for all $j \in \{1, 2\}$. Thus, we have $\beta_1 \lesssim_{\Psi_i} \beta_2$, contrary to the assumption. □

Lemma 5.5. *Let I be an abelian nilradical of Φ^+, Ψ a parabolic root subsystem of Φ, and Π_Ψ the simple system of Ψ^+. Assume that the following conditions hold:*

(a) $\Pi_\Psi \smallsetminus I \subseteq \Pi$*;*
(b) *if Ψ', Ψ'' are distinct irreducible components of Ψ, then $I \cap \Psi'$ and $I \cap \Psi''$ are element-wise incomparable.*

Then $I \cap \Psi$ is saturated, hence $\sim$closed.

Proof. Let $\gamma_1, \gamma_2 \in I \cap \Psi$ with $\gamma_1 < \gamma_2$. Then $\gamma_2 - \gamma_1$ is a linear combination of roots in Π with non-negative, integral, coefficients.

Let $\Psi_1, \dots, \Psi_k$ be the irreducible components of Ψ and Π_{Ψ_i} the simple system of Ψ_i, for $i = 1, \dots, k$. By assumption (b), there exists $i \in \{1, \dots, k\}$ such that $\gamma_1, \gamma_2 \in \Psi_i$.

By Proposition 3.5, $I \cap \Psi_i \neq \varnothing$ if and only if there exists $\beta_i^* \in I \cap \Pi_{\Psi_i}$. Moreover, in such a case, we have $I \cap \Pi_{\Psi_i} = \{\beta_i^*\}$, and $I \cap \Psi_i = \Psi_i^+ \setminus \Phi(\Pi_{\Psi_i} \setminus \{\beta_i^*\})$. It follows that $\gamma_2 - \gamma_1$ is a $\mathbb{Z}$-linear combination of roots in $\Pi_{\Psi_i} \setminus \beta_i^*$.

By assumption (a), $\Pi_{\Psi_i} \setminus \beta_i^* \subseteq \Pi$, hence, being Π a linear basis of E, $\gamma_2 - \gamma_1$ is a linear combination of roots in $\Pi_{\Psi_i} \setminus \beta_i^*$ with non-negative integral coefficients. Since Ψ, and hence Ψ_i, is parabolic, we obtain that all $\gamma \in \Phi$ such that $\gamma_1 \leqslant \gamma \leqslant \gamma_2$ belong to Ψ_i, hence to $I \cap \Psi_i$.

This proves that $I \cap \Psi$ is saturated and hence, by Lemma 5.3, also $\sim$closed. □

We recall that we have defined the set $\mathrm{Red}(\beta)$, for $\beta \in \Phi^+$, in Definition 4.12.

Definition 5.6. *Let $J \subseteq I$, and $\beta \in J$. We say that β is a detachable element in J if the following conditions hold:*

(1) *β is an extremal element of J (with respect to the standard partial order);*
(2) *there exists a hyperplane H such that:*
   (a) *$J \cap \mathrm{Red}(\beta) = J \cap H$ and H strictly separates β from $J \setminus (\{\beta\} \cup \mathrm{Red}(\beta))$;*

   (b) *$I \cap H$ is $\sim$closed.*

   We call such a hyperplane H a detaching hyperplane for β in J.

Lemma 5.7. *Let I be a facet ideal, $\beta \in I \cap \Phi_\ell$, and $I(\beta^{\lesseqgtr}) = \{\gamma \in I \mid \gamma \leqslant \beta \text{ or } \beta \leqslant \gamma\}$. Then, there exists a hyperplane H such that $I \cap H$ is $\sim$closed and $I(\beta^{\lesseqgtr}) \cap \mathrm{Red}(\beta) = I \cap H$.*

Moreover, if J is such that $J \subseteq I(\beta^{\lesseqgtr})$, and β is an extremal element of J, then H is a detaching hyperplane for β in J.

Proof. By Remark 4.13, $I(\beta^{\lesseqgtr}) \cap \mathrm{Red}(\beta) = \{\gamma \in I \mid (\gamma, \beta^\vee) = 1\}$. Let α_I be the (unique) simple root such that $I = \{\gamma \in \Phi \mid c_{\alpha_I}(\gamma) = m_{\alpha_I}\}$. Recall that $\check{\omega}_{\alpha_I}$ is the fundamental coweight such that $(\alpha_I, \check{\omega}_{\alpha_I}) = 1$ and $(\alpha, \check{\omega}_{\alpha_I}) = 0$ for all $\alpha \in \Pi \setminus \{\alpha_I\}$. We set $\nu = m_{\alpha_I}\beta^\vee - \check{\omega}_{\alpha_I}$ and $H = \nu^\perp$. Then, for all $\gamma \in I$, we have $(\nu, \gamma) = 0$ if and only if $(\beta^\vee, \gamma) = 1$, hence $I \cap H = I(\beta^{\lesseqgtr}) \cap \mathrm{Red}(\beta)$.

Now, we prove that $I \cap H$ is $\sim$closed. Let $\beta_1, \beta_2 \in \Phi^+ \cap H$, $\beta_1 \sim \beta_2$, and $\{\gamma_1, \gamma_2\}$ be a middle pair between β_1 and β_2. Then $(\gamma_1 + \gamma_2, \beta^\vee) = (\beta_1 + \beta_2, \beta^\vee) = 2$. Since β is long, this forces $(\gamma_1, \beta^\vee) = (\gamma_2, \beta^\vee) = 1$, hence $\{\gamma_1, \gamma_2\} \subseteq I \cap H$, which implies the claim.

It remains to prove the second assertion. Let $J \subseteq I(\beta^{\lesseqgtr})$ and β be an extremal element in J. Then, by the previous part, $J \cap \mathrm{Red}(\beta) = J \cap H$. In order to prove that H is a detaching hyperplane for β in J, it remains to prove that H strictly separates β from $J \setminus (\{\beta\} \cup H)$. We prove that H strictly separates β from $I \setminus (\{\beta\} \cup H)$, which implies the claim. Since I is abelian, we have $(\beta^\vee, \gamma) \geqslant 0$ for all $\gamma \in I$. Moreover, we have $(\beta^\vee, \gamma) \in \{0, \pm 1\}$, since $\beta \in \Phi_\ell$. Therefore, for all $\gamma \in I \setminus H$, we have $(\beta^\vee, \gamma) = 0$, hence $(\nu, \gamma) = -m_{\alpha_I}$. Moreover, we have $(\beta, \nu) = m_{\alpha_I}$, hence the claim is proved. □

Definition 5.8. *Let $\preccurlyeq$ be a total order relation on I, and*

$$S_{I,\preccurlyeq} = \{\beta \in I \mid \mathrm{rk}(\beta^{\preccurlyeq}) = n\}.$$

We say that $\preccurlyeq$ is a triangulation order *if the following conditions hold:*

(1) *$I \setminus S_{I,\preccurlyeq}$ is saturated;*
(2) *for each $\beta \in S_{I,\preccurlyeq}$, $(\beta^{\preccurlyeq})$ is saturated and either one of the following conditions holds:*
   (a) *β is detachable in $(\beta^{\preccurlyeq})$,*

(b) $(\beta^{\preccurlyeq})$ *has a bipartition* $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ *such that, for both* $J = J_{\mathrm{i}}$ *and* $J = J_{\mathrm{f}}$, β *is detachable in* J.

Remark 5.9.

(1) The definition directly implies that, for any total ordering $\preccurlyeq$ on I, the subset $S_{I,\preccurlyeq}$ is an initial section of the ordered set $(I, \preccurlyeq)$. Moreover, we have $\mathrm{rk}(I \smallsetminus S_{I,\preccurlyeq}) < n$.
(2) The set $I \smallsetminus S_{I,\preccurlyeq}$ may be properly contained in $I \cap \mathrm{Span}(I \smallsetminus S_{I,\preccurlyeq})$. For the triangulation orders that we will construct, this happens exactly in one case, namely for type E_7.
(3) The above definition does not contain any condition on the restriction of $\preccurlyeq$ to $I \smallsetminus S_{I,\preccurlyeq}$. Hence, if $\preccurlyeq$ is a triangulation order, any other total order $\preccurlyeq'$ such that $S_{I,\preccurlyeq'} = S_{I,\preccurlyeq}$, and $\preccurlyeq$ and $\preccurlyeq'$ coincide on the initial section $S_{I,\preccurlyeq}$, is a triangulation order, too.

We will prove the existence of triangulation orders for all facet ideals. The proof requires a case by case analysis. By Proposition 3.2, we may restrict the analysis to the abelian nilradicals.

Definition 5.10. *We say that the facet ideal* I *of* Φ^+ *is an abelian nilradical of type* $\mathrm{X}_{n,k}$, *and we write* $I \cong \mathrm{X}_{n,k}$, *if there exists an irreducible root subsystem* Ψ *of* Φ *and a positive system* $\widetilde{\Psi}^+$ *of* Ψ *such that:*

(1) I *is an abelian nilradical of* $\widetilde{\Psi}^+$*;*
(2) Ψ *is of type* X_n*;*
(3) *if* $\{\alpha'_1, \dots, \alpha'_n\}$ *is a simple system of* $\widetilde{\Psi}^+$, *numbered according to Bourbaki's conventions* [2], *then* I *is the principal ideal generated by* α'_k *in* $\widetilde{\Psi}^+$.

It is implicit in the definition that the above α'_k has multiplicity 1 in Ψ.

We note that the type of a facet ideal may be not unique, if the root system Ψ has nontrivial Dynkin diagram automorphisms. We identify the types $\mathrm{X}_{n,k}$ and $\mathrm{X}_{n,k'}$ if there exists a diagram automorphism that maps α_k into $\alpha_{k'}$. By a direct inspection of the highest root in all root types, we see that the possible abelian nilradicals types, in an irreducible root system of rank n, are the following:

$$\mathrm{A}_{n,k} \text{ for } k = 1, \dots, n, \quad \mathrm{B}_{n,1}, \quad \mathrm{C}_{n,n}, \quad \mathrm{D}_{n,k} \text{ for } k = 1, n-1, n, \quad \mathrm{E}_{6,1}, \quad \mathrm{E}_{6,6}, \quad \mathrm{E}_{7,7}.$$

Among them, we have the identifications: $\mathrm{A}_{n,k} = \mathrm{A}_{n,k'}$ for $k + k' = n + 1$; $\mathrm{D}_{n,n-1} = \mathrm{D}_{n,n}$ for all $n \geqslant 4$ and $\mathrm{D}_{4,1} = \mathrm{D}_{4,3} = \mathrm{D}_{4,4}$; $\mathrm{E}_{6,1} = \mathrm{E}_{6,6}$.

By Proposition 3.2, the facet ideals that are not abelian nilradicals of Φ^+ are in any case abelian nilradicals of some type. Their type $\mathrm{X}_{n,k}$ is explicitly obtained as follows.

If the type of Φ is A_n, all the facet ideals are nilradicals of Φ^+, hence we may assume that the extended Dynkin graph of Φ is a tree. Then, $I_{\mathrm{F}_{\alpha_i}}$ is a facet ideal of Φ^+ if and only if α_i is a leaf in the extended Dynkin graph. By Corollary 2.9, the Dynkin graph obtained by removing α_i from the extended Dynkin graph of Φ, gives the root type X_n. Moreover, the position of $-\theta$ in the new Dynkin graph gives the index k of the abelian nilradical type $\mathrm{X}_{n,k}$. Below, we write the resulting type for the facet ideals that are not abelian nilradicals of Φ^+ itself. If the root type of Φ is Y_n, we write $I_F(\mathrm{Y}_n, \alpha_i)$ in place of $I_{F_{\alpha_i}}$.

$$I_F(\mathrm{B}_n, \alpha_n) \cong \mathrm{D}_{n,n}, \qquad I_F(\mathrm{F}_4, \alpha_4) \cong \mathrm{B}_{4,1}, \qquad I_F(\mathrm{E}_7, \alpha_2) \cong \mathrm{A}_{7,1},$$
$$I_F(\mathrm{E}_8, \alpha_1) \cong \mathrm{D}_{8,1}, \qquad I_F(\mathrm{E}_8, \alpha_2) \cong \mathrm{A}_{8,1}.$$

In proving the next proposition, we will consider, case by case, the seven possible distinct sporadic or classes of abelian nilradical types. The main points of the proof are illustrated in Figures 1–9. We first give some explanation of these figures. We may arrange the roots of any facet ideal I in a matrix $(\beta_{i,j})$, in such a way that adjacent entries differ by a simple root. The label i on a certain edge means that the difference between its vertexes is the simple root α_i. We choose the matrix arrangement of roots so that the standard partial order is compatible with the reverse lexicographic order of row and column indexes, starting from $\beta_{1,1} = \theta$. In this way, the matrix yields a Hasse diagram of I in which the order ascends toward northwest. We note that this condition does not determine a unique possibility. The figures illustrate the proof on such Hasse diagrams for all the abelian nilradicals.

Proposition 5.11. *Each facet ideal has a triangulation order.*

Proof. By the above discussion, we may assume that I is an abelian nilradical of Φ^+.

By Remark 5.9, it suffices to define a subset $S_{I,\preccurlyeq}$ of I and a partial order $\preccurlyeq$ on I that is total on $S_{I,\preccurlyeq}$ and has $S_{I,\preccurlyeq}$ as an initial section, in such a way that all conditions of Definition 5.8 are satisfied. (In figures 1-9, the circled nodes correspond to the elements in $S_{I,\preccurlyeq}$.)

Henceforward, we write S_I in place of $S_{I,\preccurlyeq}$ and we intend that S_I is an initial section of $\preccurlyeq$. We will define the restriction $(S_I, \preccurlyeq)$ as a sequence $(\beta_1, \dots, \beta_k)$ (so that $(\beta_i^{\preccurlyeq}) = \{\beta_i, \dots, \beta_k\} \cup (I \setminus S_I))$. Then, we will find a hyperplane H_I such that $\mathrm{Span}(I \setminus S_I) = H_I$ and $\beta_k \notin H_I$: this ensures that S_I is well defined, i.e. $S_I = \{\beta \in I \mid \mathrm{rk}(\beta^{\preccurlyeq}) = n\}$. Moreover, we will prove that $I \setminus S_I$ is saturated (condition (1) of Definition 5.8).

In all cases, the sequence $(\beta_1, \dots, \beta_k)$ will be such that, for $i = 1, \dots, k$, β_i is extremal in $(\beta_i^{\preccurlyeq})$, with respect to the standard partial order. Since $I \setminus S_I$ is saturated, this implies that $(\beta_k^{\preccurlyeq})$ is saturated and, by induction on $k-i$, that $(\beta_i^{\preccurlyeq})$ is saturated, for $i = 1, \dots, k$. Therefore, in order to prove condition (2) of Definition 5.8, it will remain to prove that either condition (a), or (b), holds for all β_i.

If β_i is long and $\beta_i = \min(\beta_i^{\preccurlyeq})$, or $\beta_i = \max(\beta_i^{\preccurlyeq})$ (with respect to the standard partial order), then β_i is detachable in $(\beta_i^{\preccurlyeq})$ by Lemma 5.7 applied with $J = (\beta_i^{\preccurlyeq})$, and we have nothing to prove. In the remaining cases, we will directly prove that conditions (a) or (b) of Definition 5.8(2) hold.

Finally, since β_i is extremal in $(\beta_i^{\preccurlyeq})$, in order to prove that β_i is detachable in $(\beta_i^{\preccurlyeq})$, or in a subset of its, it will suffice to check that condition (2) of Definition 5.6 holds.

Now we can give the details of the proof for each abelian nilradical. Throughout the rest of the proof, we use the following notation: for $h, k \in \mathbb{Z}$, $[h, k]$ is the interval $\{i \in \mathbb{Z} \mid h \leqslant i \leqslant k\}$. For all $h \in [1, n]$ and $S \subseteq [1, n]$, $\check{\omega}_h = \check{\omega}_{\alpha_h}$ and $\alpha_S = \sum\limits_{i \in S} \alpha_i$.

A. $I \cong \mathrm{A}_{n,k}$, $\left[\frac{n}{2}\right] < k \leqslant n$. We recall that $\Phi^+ = \{\alpha_{[i,j]} \mid 1 \leqslant i \leqslant j \leqslant n\}$, whence $I = \{\alpha_{[i,j]} \mid 1 \leqslant i \leqslant k \leqslant j \leqslant n\}$.

We define $(S_I, \preccurlyeq) = \left(\alpha_{[k,j]} | j = k, \dots, n\right)$. Then, $I \setminus S_I$ is the principal ideal $(\alpha_{[k-1,k]}{}^{\leqslant})$ of Φ^+, in particular is saturated. Let $H_I = (\check{\omega}_k \setminus \check{\omega}_{k-1})^{\perp}$. It is easily seen that $\mathrm{Span}(I \setminus S_I) = H_I$ and $\alpha_{[k,n]} \notin H_I$, hence S_I is well defined.

It remains to prove that β is detachable in $(\beta^{\preccurlyeq})$, for all $\beta \in S_I$.

Let $\beta = \alpha_{[k,j]}$ $(j \in [k, n])$ and $H = (\check{\omega}_k - \check{\omega}_{k-1} - \check{\omega}_{j+1})^{\perp}$ (where $\check{\omega}_{n+1} = \underline{0}$). It is easy to check that β is minimal in $(\beta^{\preccurlyeq})$. We prove that H is a detaching hyperplane for β in $(\beta^{\preccurlyeq})$. Let $\Pi_1 = \left\{\alpha_1, \dots, \alpha_{k-2}, \alpha_{[k-1,k]}, \alpha_{k+1} \dots, \alpha_j\right\}$ and, if $j < n$, $\Pi_2 = \left\{\alpha_{[k,j+1]}, \alpha_{j+2} \dots, \alpha_n\right\}$, while, if $j = n$, $\Pi_2 = \varnothing$. Then, $\Pi_1 \sqcup \Pi_2$ is the simple system of $(\Phi \cap H)^+$, and $\Phi \cap H = \Phi(\Pi_1) \sqcup \Phi(\Pi_2)$ is a decomposition into irreducible components (both of type A). We have that $I \cap \Phi(\Pi_1)$ is the principal ideal generated by $\alpha_{[k-1,k]}$ in $\Phi(\Pi_1)$. Similarly, for $j < n$, $I \cap \Phi(\Pi_2)$ is the principal ideal generated by $\alpha_{[k,j+1]}$

in $\Phi(\Pi_2)$. It follows easily that $I\cap\Phi(\Pi_1)$ and $I\cap\Phi(\Pi_2)$ are pair-wise imcomparable. Hence, we may apply Lemma 5.5 (with $\Psi=\Phi\cap H$) and obtain that $I\cap H$ is $\sim$closed.

It remains to check that condition (2a) of Definition 5.6 holds. Let $\gamma\in(\beta^{\preccurlyeq})$. Looking at Π_1 and Π_2, we see that, if $\gamma\in H$, then either γ and β are incomparable for the standard partial order, or $\gamma-\beta\in\Phi^+$. If $\gamma\notin H$, then $\gamma=\alpha_{[i,i']}$ with $i<k$ and $i'<j$, hence $\gamma\geqslant\beta$ and $\gamma-\beta\notin\Phi$. By Corollary 4.10, we obtain that $\gamma\sim\beta$ if and only if $\gamma\notin H$, which is the claim.

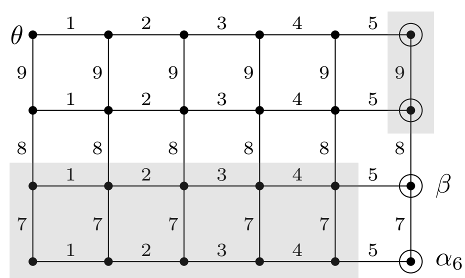


FIGURE 1. $I\cong \mathrm{A}_{9,6}$, $\beta=\alpha_{[6,7]}$. The gray boxes cover the roots in $H=(\check\omega_6-\check\omega_5-\check\omega_8)^\perp$.

C. $I\cong \mathrm{C}_{n,n}$. We recall that $\Phi^+=\{\alpha_{[i,j]},\ \alpha_{[i,n]}+\alpha_{[j,n-1]}\mid 1\leqslant i\leqslant j\leqslant n\}$, hence $I=\{\alpha_{[i,n]}+\alpha_{[j,n-1]}\mid 1\leqslant i\leqslant j\leqslant n\}$ (with $\alpha_{[n,n-1]}=\underline{0}$).

We define $(S_I,\preccurlyeq)=(\alpha_{[i,n]}|i=n,n-1,\dots,1)$. Then, $I\smallsetminus S_I$ is the principal ideal $(\alpha_n+2\alpha_{n-1}{}^{\leqslant})$ of Φ^+, in particular it is saturated. Let $H_I=(2\check\omega_n-\check\omega_{n-1})^\perp$. It is easily seen that $\mathrm{Span}(I\smallsetminus S_I)=H_I$ and $\alpha_{[1,n]}\notin H_I$, hence S_I is well defined.

Now, we prove that β is detachable in $(\beta^{\preccurlyeq})$, for all $\beta\in S_I$.

Let $\beta=\alpha_{[j,n]}$, with $j\in[1,n]$, $H=(2\check\omega_n-\check\omega_{n-1}-\check\omega_{j-1})^\perp$. It is easy to check that β is minimal in $(\beta^{\preccurlyeq})$. We prove that H is a detaching hyperplane. Let $\Pi_1=\{\alpha_n+2\alpha_{n-1}\}\cup\{\alpha_i\mid j\leqslant i\leqslant n-2\}$, if $j\leqslant n-1$, and $\Pi_1=\varnothing$ if $j=n$. Moreover, let $\Pi_2=\{\alpha_{[j-1,n]}\}\cup\{\alpha_i\mid 1\leqslant i\leqslant j-2\}$, if $j\leqslant 2$, and $\Pi_2=\varnothing$ for $i=1$. Then, $\Pi_1\sqcup\Pi_2$ is a simple system for $\Phi\cap H$. Moreover, $\Phi\cap H=\Phi(\Pi_1)\cup\Phi(\Pi_2)$ is a decomposition into irreducible components (of type C and A, respectively). It is easily seen that $I\cap\Phi(\Pi_1)$ and $I\cap\Phi(\Pi_2)$ are element-wise incomparable, hence, the conditions of Lemma 5.5 are satisfied, with $\Psi=\Phi\cap H$. It follows that $I\cap H$ is $\sim$closed.

If $\gamma\in I$, then $\gamma=\alpha_{[h,n]}+\alpha_{[k,n-1]}$ for some $1\leqslant h\leqslant k\leqslant n$. Hence, $\gamma\in H$ if and only if either $h\leqslant j-1$ and $k=n$, or $j\leqslant h\leqslant k\leqslant n-1$. In these cases, either γ and β are incomparable for the standard partial order, or $\gamma-\beta\in\Phi^+$. Moreover, if $\gamma-\beta\in\Phi^+$, then all γ' such that $\gamma<\gamma'<\beta$ are short roots. Hence, in any case, $\gamma\not\sim\beta$, by Proposition 4.9(3). If $\gamma\in(\beta^{\preccurlyeq})\smallsetminus H$, we have $\gamma=\alpha_{[h,n]}+\alpha_{[k,n-1]}$ with $h\leqslant j-1\leqslant k\leqslant n-1$. Then, $\beta+\alpha_{[k,n-1]}\in\Phi$ and we obtain the crossing relation $\beta+\gamma=(\beta+\alpha_{[k,n-1]})+\alpha_{[h,n]}$. It follows that H satisfies the conditions of Definition 5.6, hence β is detachable in $(\beta^{\preccurlyeq})$ (see Figure 2).

B1 and **D1.** $I\cong \mathrm{B}_{n,1}$, or $I\cong \mathrm{D}_{n,1}$. In case $\mathrm{B}_{n,1}$, we have

$$I=\{\alpha_{[1,i]}\mid 1\leqslant i\leqslant n\}\cup\{\alpha_{[1,n]}+\alpha_{[j,n]}\mid 2\leqslant j\leqslant n\}.$$

In case $\mathrm{D}_{n,1}$, we have

$$I=\{\alpha_{[1,i]}\mid 1\leqslant i\leqslant n\}\cup\{\widehat\alpha_{[1,n]}\}\cup\{\alpha_{[1,n]}+\alpha_{[j,n]}\mid 2\leqslant j\leqslant n-2\},$$

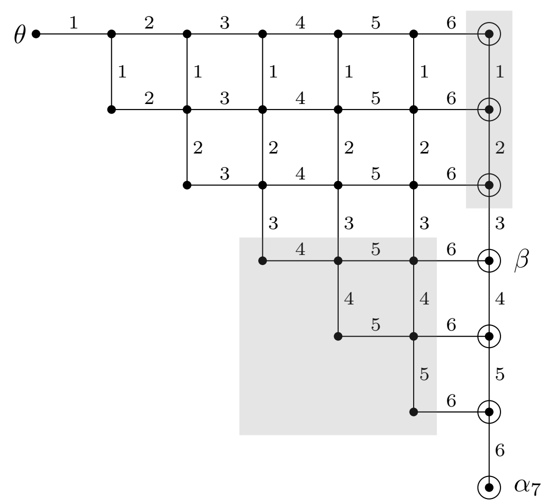


Figure 2. $I \cong \mathrm{C}_{7,7}$, $\beta = \alpha_{[4,7]}$. The gray boxes cover the roots in $H = (2\check{\omega}_7 - \check{\omega}_3 - \check{\omega}_6)^{\perp}$.

where $\widehat{\alpha}_{[1,n]} = \alpha_{[1,n]} - \alpha_{n-1}$.

We define $(S_I, \preccurlyeq) = (\alpha_1, \theta)$. Let $H_I = (\check{\omega}_2 - \check{\omega}_1)^{\perp}$. Then $\mathrm{Span}(I \smallsetminus S_I) = H_I$ and $\theta \notin H_I$, hence S_I is well defined. Since S_I consists of the minimum and the maximum of I, we have that $I \smallsetminus S_I$ is saturated.

Finally, for each $\beta \in S_I$, β is either the minimum, or the maximum of $(\beta^{\preccurlyeq})$, with respect to the standard partial order, hence β is detachable in $(\beta^{\preccurlyeq})$ by Lemma 5.7.

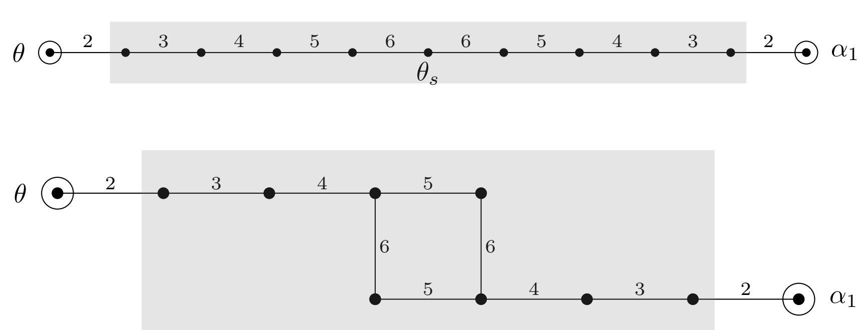


Figure 3. $I \cong \mathrm{B}_{6,1}$ and $I \cong \mathrm{D}_{6,1}$. In both cases, $S_I = \{\alpha_1, \theta\}$. The gray boxes cover the roots in $I \smallsetminus S_I = I \cap H$, with $H = (\check{\omega}_1 - \check{\omega}_2)^{\perp}$.

Dn. $I \cong \mathrm{D}_{n,n}$. Let $\widehat{\alpha}_{[j,n]} = \alpha_{[j,n]} - \alpha_{n-1}$, for $1 \leqslant j \leqslant n-1$. Then,

$$I = \{\widehat{\alpha}_{[j,n]} \mid 1 \leqslant j \leqslant n-1\} \cup \{\alpha_{[j,n]} \mid 1 \leqslant j \leqslant n-2\} \cup \{\alpha_{[i,n]} + \alpha_{[j,n-2]} \mid 1 \leqslant i < j \leqslant n-2\}.$$

We define $(S_I, \preccurlyeq) = (\widehat{\alpha}_{[j,n]} | j = n, n-2, \dots, 1)$ and take $H_I = (\check{\omega}_n \smallsetminus \check{\omega}_{n-1})^{\perp}$. Then, $I \smallsetminus S_I$ is the principal ideal $(\alpha_{[n-2,n]}{}^{\leqslant})$ of Φ^+, hence it is saturated. Moreover, we have $\mathrm{Span}(I \smallsetminus S_I) = H_I$ and $\widehat{\alpha}_{[1,n]} \notin H_I$, hence S_I is well defined.

It remains to prove that, for all $\beta \in S_I$, either one or the other of conditions (a) and (b) of Definition 5.8 (2) is satisfied. If $\beta = \alpha_n$ or $\beta = \alpha_n + \alpha_{n-2}$, then β is the minimum of $(\beta^{\preccurlyeq})$, hence it is detachable in $(\beta^{\preccurlyeq})$ by Lemma 5.7. Then, let $\beta = \widehat{\alpha}_{[j,n]}$, with $j \in \{1, \dots, n-3\}$. Let $J_{\mathrm{f}} = (\beta^{\leqslant})$. For $j > 1$, let $\theta_{j-i} = \alpha_{[j-1,n]} + \alpha_{[j,n-2]}$, and $J_{\mathrm{i}} = (\beta^{\preccurlyeq}) \smallsetminus (\theta_{j-1}^{\leqslant})$ (see Figure 4). For $j = 1$, let $J_{\mathrm{i}} = (\beta^{\preccurlyeq})$: in this case $J_{\mathrm{f}} \subseteq J_{\mathrm{i}}$. We prove that, for $j > 1$, $(\beta^{\preccurlyeq}) = J_{\mathrm{i}} \cup J_{\mathrm{f}}$ is a proper bipartition. Indeed, we have: $(\beta^{\preccurlyeq}) = \{\gamma \in I \mid c_{\alpha_j}(\gamma) \geqslant 1 \text{ or } c_{\alpha_{n-1}}(\gamma) \geqslant 1\}$, $J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}} = \{\gamma \in I \mid c_{\alpha_j}(\gamma) = 0 \text{ and } c_{\alpha_{n-1}}(\gamma) = 1\}$, and $J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}} = \{\gamma \in I \mid c_{\alpha_j}(\gamma) = 2\}$ (note that $c_{\alpha_j}(\gamma) = 2$ forces $c_{\alpha_n-1}(\gamma) = 1$). Hence, if we set $H = (\check{\omega}_n - \check{\omega}_j)^{\perp}$, H strictly separates $J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$ from $J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}$, and $J_{\mathrm{i}} \cap J_{\mathrm{f}} = I \cap H$. Moreover, for any $\gamma_1 \in J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$ and $\gamma_2 \in J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}$ we have $c_{\alpha_j}(\gamma_2 - \gamma_1) = 2$ and $c_{\alpha_{n-1}}(\gamma_2 - \gamma_1) = 0$. This implies $\gamma_2 - \gamma_1 \notin \Phi$, hence $\gamma_1 \lesssim \gamma_2$ by Proposition 4.9(1).

By definition, we have $\beta = \min J_{\mathrm{f}}$ hence we may apply Lemma 5.7 with $J = J_{\mathrm{f}}$ and obtain that β is detachable in J_{f}.

The proof that β is also detachable in J_{i} is very similar to the proofs of cases A_n and C_n. We take $H^{\mathrm{i}} = (\check{\omega}_n - \check{\omega}_{n-1} - \check{\omega}_{j-1})^{\perp}$, with $\check{\omega}_0 = 0$. Then, for each $\gamma \in (\beta^{\preccurlyeq})$, if $\gamma \notin H^{\mathrm{i}}$ and $\gamma \neq \beta$, we have $\gamma > \beta$ and $(\gamma, \beta) = 0$. If $\gamma \in H^{\mathrm{i}}$, then either γ is incomparable with β, or $\gamma - \beta \in \Phi$. Hence, by Corollary 4.10, $\gamma \sim \beta$ if and only if $\gamma \notin H^{\mathrm{i}}$. It remains to prove that $I \cap H^{\mathrm{i}}$ is $\sim$closed. Indeed, let $\Pi_1 = \{\alpha_{[n-1,n]}\} \cup \{\alpha_i \mid j \leqslant i \leqslant n-2\}$ and $\Pi_2 = \{\widehat{\alpha}_{[j-1,n]}\} \cup \{\alpha_i \mid 1 \leqslant i \leqslant j-2\}$ ($\Pi_2 = \varnothing$ for $j = 1$). Then, $\Pi_1 \cup \Pi_2$ is a simple system for $\Phi \cap H^{\mathrm{i}}$, and $\Phi \cap H^{\mathrm{i}} = \Phi(\Pi_1) \cup \Phi(\Pi_2)$ is a decomposition into irreducible components (of types D and A, respectively, for $j > 1$. For $j = 1$ we have only the component $\Phi(\Pi_1)$, which is irreducible of type D_{n-1}). It is easy to see that $I \cap \Phi(\Pi_1)$ and $I \cap \Phi(\Pi_2)$ are pairwise incomparable, hence we may apply Lemma 5.5 with $\Psi = \Phi \cap H^{\mathrm{i}}$ and obtain that $I \cap \Psi = I \cap H^{\mathrm{i}}$ is $\sim$closed.

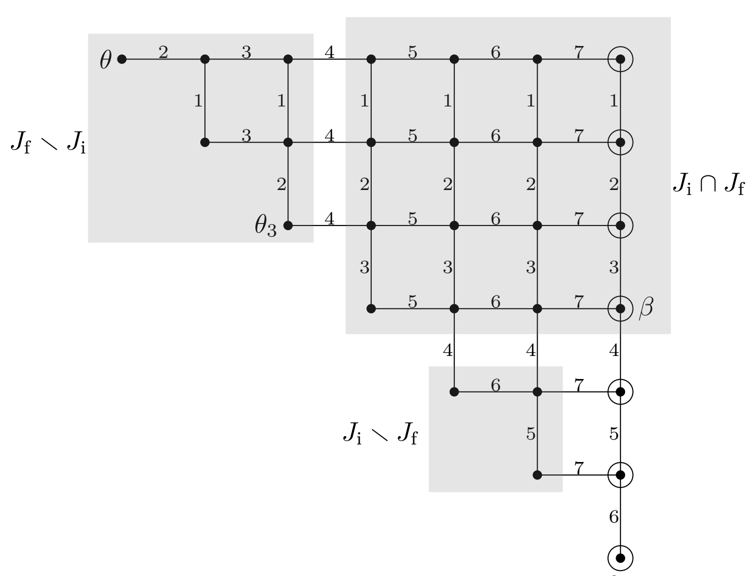


FIGURE 4. $I \cong \mathrm{D}_{8,8}$, $\beta = \widehat{\alpha}_{4,8}$. The gray boxes cover $(\beta^{\preccurlyeq})$, partitioned according to the bipartition described in the proof.

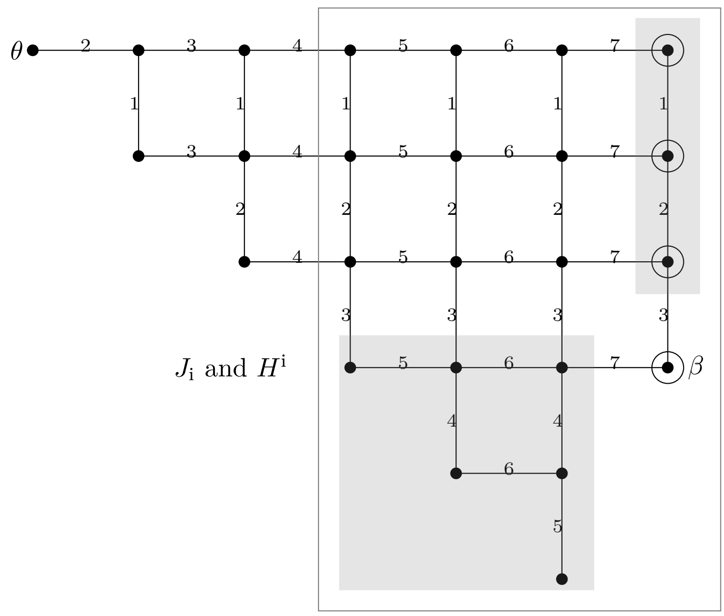


Figure 5. $I \cong \mathrm{D}_{8,8}$, $\beta = \widehat{\alpha}_{4,8}$. The diagram represents $(\beta^{\preccurlyeq})$. The big rectangle contains the roots in J_{i} and the gray parts cover the roots in $H^{\mathrm{i}} = (\breve{\omega}_8 - \breve{\omega}_7 - \breve{\omega}_3)^{\perp}$.

E6. $I \cong \mathrm{E}_{6,6}$. We choose

$$(S_I, \preccurlyeq) = \left(\alpha_6, \theta, \alpha_{\{5,6\}}, \theta - \alpha_2, \alpha_{\{4,5,6\}}, \theta - \alpha_{\{2,4\}}, \alpha_{\{2,4,5,6\}}, \theta - \alpha_{\{2,4,5\}}\right).$$

The roots in $(S_I, \preccurlyeq)$ are all the $\gamma \in I$ with $c_{\alpha_3}(\gamma) = 0$, alternated with their symmetric roots with respect to the order involution, which are all the $\gamma \in I$ with $c_{\alpha_3}(\gamma) = 2$. Hence $I \setminus S_I$ is saturated.

Let $H_I = (\breve{\omega}_6 - \breve{\omega}_3)^{\perp}$. Then, $\mathrm{Span}(I \setminus S_I) = H_I$ and $\theta - \alpha_{\{2,4,5\}} \notin H_I$, hence S_I is well defined.

Let $\Pi' = \{\alpha_{[3,6]}\} \cup \Pi \setminus \{\alpha_3, \alpha_6\}$. Then $\Phi(\Pi')$ is irreducible, of type A_5, and $I \setminus S_I$ is the abelian nilradical generated by $\alpha_{[3,6]}$ in $\Phi(\Pi')$, of type $\mathrm{A}_{5,2}$. Hence, $I \cap \mathrm{Span}(I \setminus S_I)$ is $\sim$closed by Lemma 5.5 applied with $\Psi = \Phi(\Pi')$.

The first six β in $(S_I, \preccurlyeq)$ are detachable in their $(\beta^{\preccurlyeq})$ by Lemma 5.7, being either the minimum, or the maximum of $(\beta^{\preccurlyeq})$. Hence, it remains to prove that the last two roots, $\beta = \alpha_{\{2,4,5,6\}}$ and $\beta' = \theta - \alpha_{\{2,4,5\}}$ are detachable in their $\preccurlyeq$-cone. We will prove that, in both cases, H_I is a detaching hyperplane. We have $(\beta^{\preccurlyeq}) = \{\beta\} \cup (I \setminus S_I) \cup \{\beta'\}$. It is easy to check that, for all $\gamma \in I \setminus S_I$, either β is incomparable with γ, or $\gamma - \beta \in \Phi^+$. Hence, by Corollary 4.10, $\beta \not\lesssim \gamma$ for all $\gamma \in I \setminus S_I$. For β', we have $\beta \leqslant \beta'$ and $\beta' - \beta \notin \Phi$, hence $\beta \lesssim \beta'$. Now, H_I strictly separates β from β' since $c_{\alpha_3}(\beta) = 0$ and $c_{\alpha_3}(\beta') = 2$. Morever, we have already seen that $I \setminus S_I$ is equal to $I \cap H_I$ and is saturated, hence $\sim$closed. It follows that β is detached in $(\beta^{\preccurlyeq})$, with detaching hyperplane H_I. For β', the proof is similar (see Figure 6).

E7. $I \cong \mathrm{E}_{7,7}$. We recall that $\theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. The order involution maps α_2 and α_4 onto their opposite roots, α_7 onto θ, and the sequence $(\alpha_1, \alpha_3, \alpha_5, \alpha_6)$ onto $(-\alpha_6, -\alpha_5, -\alpha_3, -\alpha_1)$.

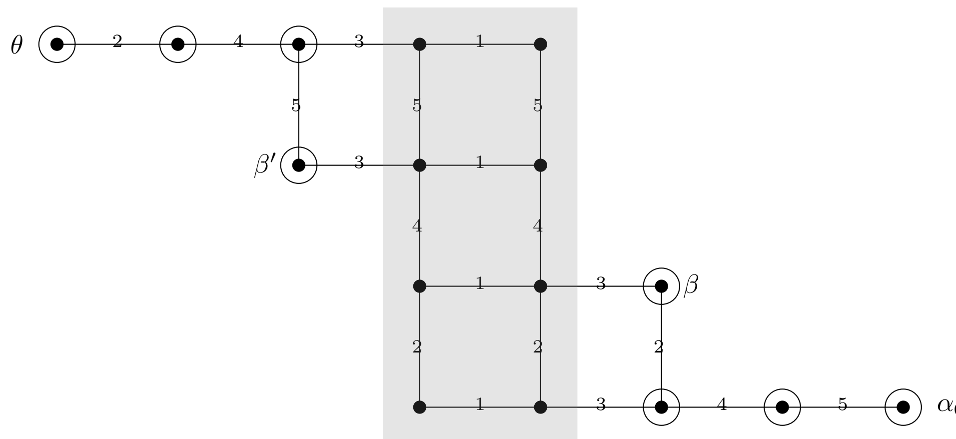


FIGURE 6. $I \cong \mathrm{E}_{6,6}$, $\beta = \alpha_{\{2,4,5,6\}}$, and $\beta' = \theta - \alpha_{\{2,4,5\}}$. The gray rectangle covers the roots in $H_I = (\check{\omega}_6 - \check{\omega}_3)^{\perp}$.

For any $\gamma \in I$, we denote by γ' the symmetric of γ with respect to the order involution and we define

$$(S_I, \preccurlyeq) = (\alpha_7,\, \alpha'_7,\, \alpha_{\{6,7\}},\, \alpha'_{\{6,7\}},\, \alpha_{\{5,6,7\}},\, \alpha'_{\{5,6,7\}},\, \alpha_{\{4,5,6,7\}},\, \alpha'_{\{4,5,6,7\}},\, \alpha_{\{2,4,5,6,7\}},\\ \alpha'_{\{2,4,5,6,7\}},\, \alpha_{\{3,4,5,6,7\}},\, \alpha'_{\{3,4,5,6,7\}},\, \alpha_{\{1,3,4,5,6,7\}},\, \alpha'_{\{1,3,4,5,6,7\}}).$$

By definition, $(S_I, \preccurlyeq)$ consists of all β in I such that $c_{\alpha_2}(\beta) + c_{\alpha_3}(\beta) \leqslant 1$, together with their symmetric roots (see Figure 7). Then, we have $\min(I \setminus S_I) = \alpha_{[2,7]}$, hence $\max(I \setminus S_I) = \alpha'_{[2,7]}$. In particular, $I \setminus S_I$ is a root interval, hence it is saturated.

Let $H_I = (\check{\omega}_7 - \check{\omega}_2)^{\perp}$ and let $\beta = \alpha_{\{2,4,5,6,7\}}$. We can directly check that $H_I \cap I = (I \setminus S_I) \cup \{\beta, \beta'\}$. We have $\beta, \beta' \in S_I$, nevertheless, $\mathrm{Span}(I \setminus S_I) = H_I$. Moreover, $\max_{\preccurlyeq} S_I = \alpha'_{\{1,3,4,5,6,7\}} = \theta - \alpha_{\{1,3,4,5,6\}} \notin H_I$. Hence, S_I is well defined (see Figure 7).

All roots in S_I, except $\beta = \alpha_{\{2,4,5,6,7\}}$, $\eta = \alpha_{\{1,3,4,5,6,7\}}$ and their symmetric roots, are either the minimum, or the maximum of their $\preccurlyeq$-upper cone, with respect to the standard partial order, so they are detachable in it by Lemma 5.7. It remains to consider $\beta, \beta', \eta, \eta'$.

First, we find a bipartition of $(\beta^{\preccurlyeq})$ that satisfies the requirements of Definition 5.8(2). Let $J_{\mathrm{f}} = (\beta^{\preccurlyeq}) \cap (\beta^{\leqslant}) = \{\gamma \in (\beta^{\preccurlyeq}) \mid c_{\alpha_2}(\gamma) \geqslant 1\}$ and $J_{\mathrm{i}} = \{\gamma \in (\beta^{\preccurlyeq}) \mid c_{\alpha_2}(\gamma) \leqslant 1\}$. It is easily seen that $J_{\mathrm{i}} \cap J_{\mathrm{f}} = I \cap H_I$ and that H_I strictly separates $J_{\mathrm{i}} \setminus J_{\mathrm{f}}$ from $J_{\mathrm{f}} \setminus J_{\mathrm{i}}$. Moreover, we can directly check that, for all $\gamma_1 \in J_{\mathrm{i}} \setminus J_{\mathrm{f}}$ and $\gamma_2 \in J_{\mathrm{f}} \setminus J_{\mathrm{i}}$, we have $\gamma_2 - \gamma_1 \in L^+(\Phi) \setminus \Phi^+$, hence $\gamma_1 \lesssim \gamma_2$. It remains to check that β is detachable in J_{i} and J_{f}. For J_{f} this follows from Lemma 5.7, since $\beta = \min_{\leqslant} J_{\mathrm{f}}$. For J_{i}, we prove that the hyperplane $H^{\mathrm{i}} = (\check{\omega}_7 - \check{\omega}_3)^{\perp}$ is a detaching hyperplane. Indeed, $(\Phi \cap H^{\mathrm{i}})$ is an irreducible root subsystem of type A_6, with simple system $\{\alpha_{[3,7]}\} \cup \Pi \setminus \{\alpha_7, \alpha_3\}$. Hence, we may apply Lemma 5.5, with $\Psi = \Phi \cap H^{\mathrm{i}}$, and find that $I \cap H^{\mathrm{i}}$ is $\sim$closed (see also Figure 8). Moreover, we can directly check that, for all $\gamma \in J_{\mathrm{i}} \cap H^{\mathrm{i}}$, either γ and β are incomparable, or $\gamma - \beta \in \Phi^+$, hence $\gamma \not\sim \beta$. Finally, for all $\gamma \in J_{\mathrm{i}} \setminus H^{\mathrm{i}}$, we have $\gamma > \beta$ and $\gamma - \beta \notin \Phi$, hence, by Corollary 4.10, $\gamma \sim \beta$. Hence H^{i} is a detaching hyperplane for β in J_{i} (see Figure 8).

The case of β' is similar to the previous one, by symmetry.

It remains to deal with η and η'. In this case, H_I is a detaching hyperplane for η in $(\eta^{\preccurlyeq})$ as well as for η' in $(\eta'^{\preccurlyeq})$ (see Figure 9). Indeed, it is easily checked that for

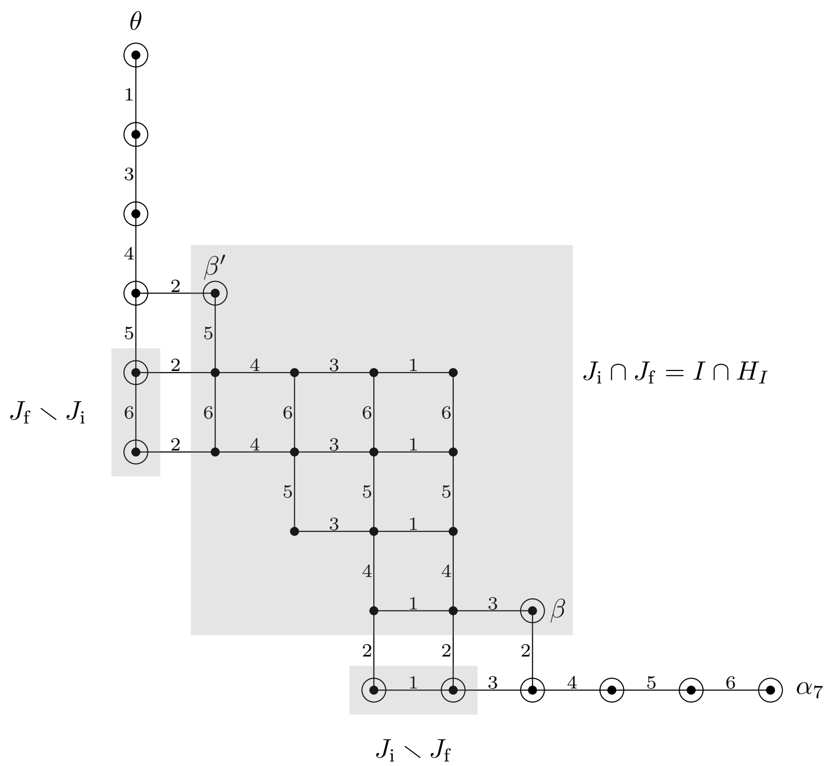


Figure 7. $I \cong \mathrm{E}_{7,7}$, $\beta = \alpha_{\{2,4,5,6,7,\}}$. The gray rectangles illustrate the bipartition of $(\beta^{\preccurlyeq})$. For the symmetric root $\beta' = \theta - \alpha_{[1,4]}$, the bipartition of $(\beta'^{\preccurlyeq})$ is similar.

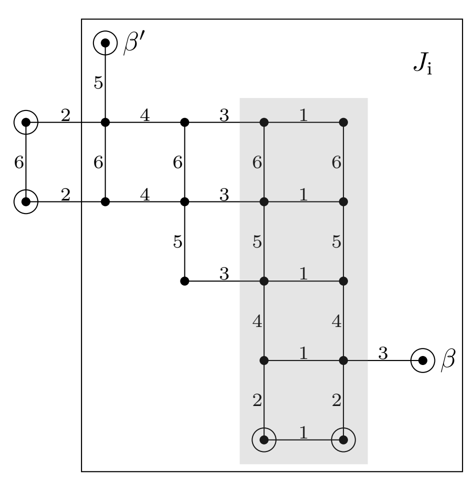


Figure 8. $I \cong \mathrm{E}_{7,7}$, $\beta = \alpha_{\{2,4,5,6,7,\}}$. The diagram represents $(\beta^{\preccurlyeq})$. The big rectangle contains the roots in J_{i} and the gray part covers the roots in $H^{\mathrm{i}} = (\breve{\omega}_7 - \breve{\omega}_3)^{\perp}$.

all $\gamma \in (\eta^{\preccurlyeq}) \cap H_I$, either γ is incomparable with η and η', or $\gamma - \eta$, $\gamma - \eta' \in \Phi$. Hence $\gamma \not\sim \eta, \eta'$. Moreover, $(\eta^{\preccurlyeq}) \smallsetminus H_I = \{\eta, \eta'\}$, H_I separates η from η', and $\eta \sim \eta'$. Finally, $(\eta'^{\preccurlyeq}) \smallsetminus H_I = \{\eta'\}$. This concludes the proof. $\square$

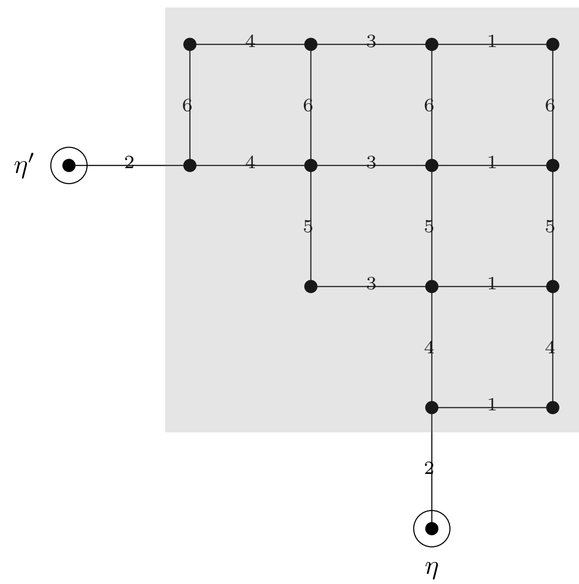


Figure 9. $I \cong \mathrm{E}_{7,7}$, $\gamma = \alpha_{\{1,3,4,5,6,7,\}}$. The diagram represents $(\eta^{\preccurlyeq})$. The gray square covers the roots contained in $H_I = (\check{\omega}_7 - \check{\omega}_2)^{\perp}$.

6. Triangulations of standard parabolic facets

In this section we prove Theorems 1.1 and 1.2.

Let I be a face ideal of Φ^+ and $F_I = \mathrm{Conv}(I)$ be the corresponding standard parabolic face. For all $J \subseteq I$, let

$$\mathcal{R}_J = \{R \subseteq J \mid R \text{ reduced}\}.$$

Then, let

$$\mathcal{T}_I = \{\mathrm{Conv}(R) \mid R \in \mathcal{R}_I, R \text{ maximal in } \mathcal{R}_I\}.$$

We will prove that $\mathcal{T}_I$ is a triangulation of F_I.

By Propositions 3.1 and 3.2, it suffices to prove the claim when I is an abelian nilradical of Φ^+. Henceforward, we make this assumption. So, there exists a unique simple root $\alpha_I \in \Pi$ such that $m_{\alpha_I} = 1$ and $I = (\alpha_I^{\leqslant})$. In particular, I is a facet ideal and F_I is a facet of $\mathcal{P}$.

The proof is by induction on $\mathrm{rk}(\Phi)$ and is based on the existence of triangulation orders for all facet ideals. We start with two key lemmas.

For each $J \subseteq I$ let $\mathrm{Cone}(J)$ be the positive cone generated by J, i.e. the set of linear combinations of elements in J with nonnegative real coefficients. Moreover, let $[J]$ be the *saturation* of J, i.e.

$$[J] = \{x \in I \mid \exists\, y, z \in J \;\; y \leqslant x \leqslant z\}.$$

Lemma 6.1. *Let J be a saturated subset of I, and $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ be a bipartition of J. Then* $\mathrm{Cone}(J) = \mathrm{Cone}(J_{\mathrm{i}}) \cup \mathrm{Cone}(J_{\mathrm{f}})$.

Proof. The claim is obvious if the bipartition is not proper, in particular if $|J| \leqslant 2$. The inclusion $\mathrm{Cone}(J_{\mathrm{i}}) \cup \mathrm{Cone}(J_{\mathrm{f}}) \subseteq \mathrm{Cone}(J)$ is clear in all cases. We prove the reverse inclusion by induction on $|J|$. It is easily seen that, for any $K \subseteq J$, $\{K \cap J_{\mathrm{i}}, K \cap J_{\mathrm{f}}\}$ is a bipartition of K. Therefore, it suffices to prove that if the bipartition $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ of J is proper, there exists a proper saturated subset K of J such that $x \in \mathrm{Cone}(K)$.

So, let $J_{\mathrm{i}}, J_{\mathrm{f}} \subsetneqq J$, $x \in \mathrm{Cone}(J)$, and $x = \sum\limits_{\beta \in J} c_\beta \beta$, with c_β nonnegative real coefficients, be a fixed expression of x. If $\{\beta \in J \mid c_\beta > 0\}$ is included in J_{i} or J_{f} we are done. Also if the saturation $[\{\beta \in J \mid c_\beta > 0\}]$ is properly included in J we are done. Hence, we assume $J = [\{\beta \in J \mid c_\beta > 0\}]$. This means that, for all $\beta \in \mathrm{Min}\, J \cup \mathrm{Max}\, J$, $c_\beta > 0$. Since $J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}$ and $J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}$ are an initial and a final section of J, we have $\mathrm{Min}(J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}) \subseteq \mathrm{Min}\, J$ and $\mathrm{Max}(J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}) \subseteq \mathrm{Max}\, J$. We fix $\beta_1 \in \mathrm{Min}(J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}})$ and $\beta_2 \in \mathrm{Max}(J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}})$. By Definition 5.1, and since J is saturated, there exist $\gamma_1, \gamma_2 \in J$ such that $\beta_1 + \beta_2 = \gamma_1 + \gamma_2$ and $\beta_1 < \{\gamma_1, \gamma_2\} < \beta_2$. Hence, $c_{\beta_1}\beta_1 + c_{\beta_2}\beta_2 = (c_{\beta_1} - c_{\beta_2})\beta_1 + c_{\beta_2}(\gamma_1 + \gamma_2) = (c_{\beta_2} - c_{\beta_1})\beta_2 + c_{\beta_1}(\gamma_1 + \gamma_2)$. We obtain that, if $c_{\beta_1} \geqslant c_{\beta_2}$, then $x \in \mathrm{Cone}(J \smallsetminus \{\beta_2\})$, while, if $c_{\beta_2} \geqslant c_{\beta_1}$, then $x \in \mathrm{Cone}(J \smallsetminus \{\beta_1\})$. Since β_1 and β_2 are extremal elements in J, $J \smallsetminus \{\beta_1\}$ and $J \smallsetminus \{\beta_2\}$ are saturated, hence the claim is proved. $\square$

Lemma 6.2. *Let J be a saturated subset of I, $\beta^* \in J$, β^* detachable in J, and $J_{\beta^*} = \{\beta^*\} \cup (\mathrm{Red}(\beta^*) \cap J)$. Then,* $\mathrm{Cone}(J) = \mathrm{Cone}(J_{\beta^*}) \cup \mathrm{Cone}(J \smallsetminus \{\beta^*\})$.

Proof. Let $J' = J \smallsetminus \mathrm{Red}(\beta^*)$. Then, $\mathrm{Cone}(J) = \mathrm{Cone}(J') + \mathrm{Cone}(\mathrm{Red}(\beta^*) \cap J) \subseteq \mathrm{Cone}(J') + \mathrm{Cone}(J_{\beta^*})$. Hence, it suffices to prove that $\mathrm{Cone}(J') \subseteq \mathrm{Cone}(J_{\beta^*}) \cup \mathrm{Cone}(J \smallsetminus \{\beta^*\})$.

Let $x \in \mathrm{Cone}(J')$, $\mathcal{K} = \{K \subseteq J' \mid x \in \mathrm{Cone}(K)\}$, and $d = \min\{|[K]| \mid K \in \mathcal{K}\}$, where $[K]$ is the saturation of K and $|[K]|$ its cardinality. Then, fix a $K \in \mathcal{K}$ with $|[K]| = d$ and let $x = \sum\limits_{\beta \in K} c_\beta \beta$, with $c_\beta \geqslant 0$, be a fixed expression of x.

If $\beta^* \notin K$, $K \subseteq J \smallsetminus \{\beta^*\}$ and we are done. Hence, let $\beta^* \in K$. We will prove that then $K = \{\beta^*\}$, which yields the claim, since $\{\beta^*\} \subseteq J_{\beta^*}$. By assumption, β^* is extremal in J hence in K. By symmetry, we may assume, without loss of generality, that β^* is minimal in K. Then, let β be a maximal element in K. If $\beta \neq \beta^*$, then, by definition of J', we have $\beta^* \lnsim \beta$, hence there exists a middle pair $\{\gamma_1, \gamma_2\}$ between β^* and β. If $c_{\beta^*} \geqslant c_\beta$, we have $c_{\beta^*}\beta^* + c_\beta\beta = (c_{\beta^*} - c_\beta)\beta^* + c_\beta(\beta^* + \beta) = (c_{\beta^*} - c_\beta)\beta^* + c_\beta(\gamma_1 + \gamma_2)$, hence $x \in \mathrm{Cone}([K] \smallsetminus \{\beta\})$. This contradicts the minimality of $|[K]|$, since $[K] \smallsetminus \{\beta\} = [K \smallsetminus \{\beta\}]$. Similarly, if $c_{\beta^*} < c_\beta$, we obtain $x \in \mathrm{Cone}([K] \smallsetminus \{\beta^*\})$, contrary to the minimality of $|[K]|$. $\square$

Proposition 6.3. *For all $J \subseteq I$, if J is saturated, then*

$$\mathrm{Cone}(J) = \bigcup \{\mathrm{Cone}(R) \mid R \subseteq J,\ R \textit{ reduced}\}.$$

Proof. The claim is obvious if $\mathrm{rk}(\Phi) = 1$. We assume $\mathrm{rk}(\Phi) \geqslant 2$ and the claim holds for any abelian nilradical in any irreducible root system of rank strictly lower than $\mathrm{rk}(\Phi)$.

Let $J \subseteq I$ be saturated. The inclusion "$\supseteq$" is clear, so it suffices to prove the reverse one. We assume $x \in \mathrm{Cone}(J)$ and prove that there exists a reduced subset R of J such that $x \in \mathrm{Cone}(R)$.

Let $\preccurlyeq$ be a triangulation order on I. We distinguish two cases.

(a) First, we consider the case $J \subseteq I \smallsetminus S_{I,\preccurlyeq}$. Let $\{c_\beta \mid \beta \in J\}$ be a fixed set of nonnegative real coefficients such that $x = \sum\limits_{\beta \in J} c_\beta \beta$. Let $\Psi = \Phi \cap \mathrm{Span}(I \smallsetminus S_{I,\preccurlyeq})$,

$\Psi_1, \dots, \Psi_k$ be the irreducible components of Ψ, $I_i = I \cap \Psi_i$, $J_i = J \cap \Psi_i$. Let $x_i = \sum\limits_{\beta \in J_i} c_\beta \beta$, for $i = 1, \dots, k$. Then, for all $i \in \{1, \dots, k\}$, I_i is an abelian nilradical of Ψ_i^+ and J_i is saturated in Ψ_i. Hence, by the induction assumption, there exists a subset R_i of J_i, reduced relatively to Ψ_i, such that $x_i \in \mathrm{Cone}(R_i)$. Let $R = R_1 \cup \dots \cup R_k$. Now, $R \subseteq J$ and J is $\sim$closed, being saturated, hence, by Lemma 5.4, R is reduced in Φ. Since $x \in \mathrm{Cone}(R)$, we are done.

(b) Now, we consider the case $J \not\subseteq I \cap S_{I,\preccurlyeq}$. Let

$$\beta_0 = \max_{\preccurlyeq}\{\beta \in J \mid x \in \mathrm{Cone}(J \cap (\beta^{\preccurlyeq}))\}.$$

If $\beta_0 \notin S_{I,\preccurlyeq}$, then $x \in \mathrm{Cone}(J \cap (I \setminus S_{I,\preccurlyeq}))$. Since $J \cap (I \setminus S_{I,\preccurlyeq})$ is saturated, being the intersection of two saturated sets, we are reduced to case (a).

Then, we assume $\beta_0 \in S_{I,\preccurlyeq}$. In this case, $({\beta_0}^{\preccurlyeq})$ is saturated (by Definition 5.8 (2)), hence, $J \cap ({\beta_0}^{\preccurlyeq})$ is saturated. It suffices to prove that there exists a reduced subset R contained in $J \cap ({\beta_0}^{\preccurlyeq})$ such that $x \in \mathrm{Cone}(R)$. By definition of triangulation order, either β_0 is detachable in $({\beta_0}^{\preccurlyeq})$, or $({\beta_0}^{\preccurlyeq})$ has a bipartition $\{B_{\mathrm{i}}, B_{\mathrm{f}}\}$ such that β_0 is a detachable element in both of B_{i} and B_{f}. In the first case, we set $B = (\beta_0)^{\preccurlyeq}$. In the latter case, $\{J \cap B_{\mathrm{i}}, J \cap B_{\mathrm{f}}\}$ is a bipartition of $J \cap ({\beta_0}^{\preccurlyeq})$ and, by Lemma 6.1, we may choose a $B \in \{B_{\mathrm{i}}, B_{\mathrm{f}}\}$ such that $x \in \mathrm{Cone}(J \cap B)$.

In both cases, we set $J' = J \cap B$. Then J' is saturated, since J and B are. Moreover, $\beta_0 = \min_{\preccurlyeq} J'$, hence, by definition of β_0, in any expression of x as a nonnegative linear combination of elements of J', the coefficient of β_0 is strictly positive. Since β_0 is detachable in B, there exists a detaching hyperplane H for β_0 in B. Then, such an H is a detaching hyperplane also for β_0 in J'. By Lemma 6.2, we obtain $\mathrm{Cone}(J') = \mathrm{Cone}(\{\beta_0\} \cup (J' \cap H)) \cup \mathrm{Cone}(J' \setminus \{\beta_0\})$, hence $x \in \mathrm{Cone}(\{\beta_0\} \cup (J' \cap H))$.

Thus, there exists a positive real c_0 such that $x - c_0\beta_0 \in \mathrm{Cone}(J' \cap H)$. Now, $J' \cap H$ is contained in the abelian nilradical $I \cap H$ of $(\Phi \cap H)^+$. Let $\Psi_1, \dots, \Psi_k$ be the irreducible components of $\Phi \cap H$. Arguing as in case (a), we find $R_1, \dots, R_k$ such that $R_i \subseteq J' \cap \Psi_i$, R_i is reduced relatively to Ψ_i, and $x - c_0\beta_0 \in \mathrm{Cone}(R_1 \cup \dots \cup R_k)$. Let $R' = R_1 \cup \dots \cup R_k$. Then, since $R' \subseteq I \cap H$ and $I \cap H$ is $\sim$closed, we have that R' is reduced in Φ, by Lemma 5.4. Moreover, by definition of β_0, we have $R' \subseteq ({\beta_0}^{\preccurlyeq})$, hence $R' \subseteq ({\beta_0}^{\preccurlyeq}) \cap H$. By Definition 5.6, $({\beta_0}^{\preccurlyeq}) \cap H \subseteq \mathrm{Red}(\beta_0)$, hence $R = \{\beta_0\} \cup R'$ is reduced. This proves the claim, since $x \in \mathrm{Cone}(R)$. ☐

Remark 6.4. For each face F_α and $J \subseteq I_{F_\alpha}$, we have $\mathrm{Cone}(J) \cap F_\alpha = \mathrm{Conv}(J)$, since $(\sum\limits_{\beta \in J} c_\beta \beta, \check{\omega}_\alpha) = \sum\limits_{\beta \in J} c_\beta (\beta, \check{\omega}_{\alpha_I}) = \sum\limits_{\beta \in J} c_\beta m_\alpha = m_\alpha$ if and only if $\sum\limits_{\beta \in J} c_\beta = 1$.

Corollary 6.5. *Let*

$$\mathcal{T}'_I = \{\mathrm{Conv}(R) \mid R \in \mathcal{R}_I,\ \mathrm{rk}(R) = n\}.$$

Then $\mathcal{T}'_I$ is a covering of F_I.

Proof. By Proposition 6.3 and the above remark, the set of all $\mathrm{Conv}(R)$, with $R \subseteq I$ and R reduced, is a covering of F_I. By standard topological arguments, we obtain that also $\mathcal{T}'_I$ is a covering of F_I. ☐

Our next step is to prove that the set $\mathcal{T}'_I$ defined in Corollary 6.5 is a triangulation of the standard parabolic facet F_I. For this, it remains to prove that each $T \in \mathcal{T}'_I$ is a simplex, and that the intersection of any two $T_1, T_2 \in \mathcal{T}_I$ is a common face of T_1 and T_2. This is proved in next two propositions.

Proposition 6.6. *Let R be a reduced subset of I. Then R is linearly independent.*

Proof. We prove the claim by induction on $\mathrm{rk}(\Phi)$. The case $\mathrm{rk}(\Phi) = 1$ is obvious. We assume $\mathrm{rk}(\Phi) > 1$ and the claim true for irreducible root systems of rank less than $\mathrm{rk}(\Phi)$. Let $\preccurlyeq$ be a triangulation order on I and $\beta = \min_{\preccurlyeq} R$.

First, we consider the case $\mathrm{rk}(\beta^{\preccurlyeq}) = n$ and β detachable in $(\beta^{\preccurlyeq})$. Let H be a detaching hyperplane for β in $(\beta^{\preccurlyeq})$, $\Psi_1, \dots, \Psi_k$ be the irreducible components of $\Phi \cap H$, and $R_i = (R \setminus \{\beta\}) \cap \Psi_i$, for $i = 1, \dots, k$. Then, R_i is contained in the abelian nilradical $I \cap \Psi_i$ of Ψ_i^+ and is reduced, relatively to Ψ_i. By the induction assumption, R_i is linearly independent, for $i = 1, \dots, k$. Since $R \setminus \{\beta\} = R_1 \cup \cdots \cup R_k$ and $\beta \notin H$, we obtain that $R \setminus \{\beta\}$ and R are linearly independent, too.

If $\mathrm{rk}(\beta^{\preccurlyeq}) = n$ and β is not detachable in $(\beta^{\preccurlyeq})$, there exists a bipartition $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ of $(\beta^{\preccurlyeq})$ such that β is a detachable element both in J_{i}, and in J_{f}. By Definition 5.1, either $R \subseteq J_{\mathrm{i}}$, or $R \subseteq J_{\mathrm{f}}$, hence we can argue as in the previous case.

If $\mathrm{rk}(\beta^{\preccurlyeq}) < n$, then R is contained in the abelian nilradical $I \cap \mathrm{Span}(I \setminus S_{I,\preccurlyeq})$, in $\Phi \cap \mathrm{Span}(I \setminus S_{I,\preccurlyeq})$ and we may argue by induction as above. □

Proposition 6.7. *Let R_1, R_2 be reduced subsets in I. Then,* $\mathrm{Conv}(R_1) \cap \mathrm{Conv}(R_2) = \mathrm{Conv}(R_1 \cap R_2)$. *In particular,* $\mathrm{Conv}(R_1) \cap \mathrm{Conv}(R_2)$ *is a common face of* $\mathrm{Conv}(R_1)$ *and* $\mathrm{Conv}(R_2)$.

Proof. By Proposition 6.6, $\mathrm{Conv}(R_i)$ is the simplex with set of vertexes R_i, for $i = 1, 2$, hence $\mathrm{Conv}(R_1 \cap R_2)$ is common face of $\mathrm{Conv}(R_1)$ and $\mathrm{Conv}(R_2)$. Hence, it suffices to prove the first statement. The inclusion $\mathrm{Conv}(R_1) \cap \mathrm{Conv}(R_2) \supseteq \mathrm{Conv}(R_1 \cap R_2)$ is clear. We prove the reverse one, by induction on $\mathrm{rk}(\Phi)$.

If $\mathrm{Cone}(R_1) \cap \mathrm{Cone}(R_2) \subseteq \mathrm{Cone}(R_1 \cap R_2)$, then, by Remark 6.4, the analogous relation for the convex hulls holds. So we work with cones.

For $\mathrm{rk}(\Phi) = 1$ the claim is obvious. Let $\mathrm{rk}(\Phi) > 1$, $\preccurlyeq$ be a fixed triangulation order on I, and $\beta = \min_{\preccurlyeq}(R_1 \cup R_2)$. We may assume $\beta \in R_1$.

(a) If $\mathrm{rk}(\beta^{\preccurlyeq}) < n$, then $R_1, R_2 \subseteq I \setminus S_{I,\preccurlyeq}$. Let $\Psi_1, \dots, \Psi_k$ be the connected components of $\Phi \cap \mathrm{Span}(I \setminus S_{I,\preccurlyeq})$, and $R_{j,i} = R_j \cap \Psi_i$, for $j = 1, 2$ and $i = 1, \dots, k$. Each $R_{j,i}$ is a reduced subset in the abelian nilradical $I \cap \Psi_i$ of Ψ_i^+, hence by the induction assumption $\mathrm{Cone}(R_{1,i}) \cap \mathrm{Cone}(R_{2,i}) \subseteq \mathrm{Cone}(R_{1,i} \cap R_{2,i})$, for each i in $\{1, \dots, k\}$. This easily implies the inclusion $\mathrm{Cone}(R_1) \cap \mathrm{Cone}(R_2) \subseteq \mathrm{Cone}(R_1 \cap R_2)$.

(b) Next, let $\mathrm{rk}(\beta^{\preccurlyeq}) = n$, β be detachable in $(\beta^{\preccurlyeq})$, H be a detaching hyperplane, and $\overline{R}_i = R_i \cap H$ for $i = 1, 2$. Let H^+ and $\overline{H^+}$, H^- and $\overline{H^-}$ be the open and closed half spaces determined by H in E. By Definition 5.6, β belong either to H^+, or to H^-. We may assume $\beta \in H^+$, without loss of generality. Then $(\beta^{\preccurlyeq}) \setminus \{\beta\} \subseteq \overline{H^-}$ and, if $\gamma \in (\beta^{\preccurlyeq}) \cap H^-$, then $\beta \sim \gamma$. It follows that, for a fixed $i \in \{1, 2\}$, either $R_i \subseteq \overline{H^+}$, or $R_i \subseteq \overline{H^-}$. This implies $\mathrm{Cone}(R_i) \cap H = \mathrm{Cone}(\overline{R}_i)$. Moreover, since we are assuming $\beta \in R_1$, we have then $R_1 \subseteq \overline{H^+}$, and $R_1 \setminus \{\beta\} = \overline{R}_1$. Now, we distinguish two subcases.

(b1) Let $\beta = \min_{\preccurlyeq} R_1 \prec \min_{\preccurlyeq} R_2$. Then, $R_2 \subseteq \overline{H^-}$, hence R_1 and R_2 are weakly separated by H. It follows easily that $R_1 \cap R_2 = \overline{R}_1 \cap \overline{R}_2$ and $\mathrm{Cone}(R_1) \cap \mathrm{Cone}(R_2) = \mathrm{Cone}(\overline{R}_1) \cap \mathrm{Cone}(\overline{R}_2)$. Arguing as in case (a), with $\Phi \cap H$ in place of $\Phi \cap \mathrm{Span}(I \setminus S_{I,\preccurlyeq})$ and $\overline{R}_i$ in place of R_i, by the induction assumption we obtain $\mathrm{Cone}(\overline{R}_1) \cap \mathrm{Cone}(\overline{R}_2) \subseteq \mathrm{Cone}(\overline{R}_1 \cap \overline{R}_2)$, and hence the claim.

(b2) Let $\beta = \min_{\preccurlyeq} R_1 = \min_{\preccurlyeq} R_2$. Then both R_1 and R_2 are contained in $\overline{H^+}$, and, for all $x \in \mathrm{Cone}(R_1) \cap \mathrm{Cone}(R_2)$, there exist $c_i \in \mathbb{R}$ and $\overline{x}_i \in \mathrm{Cone}(\overline{R}_i)$ $(i = 1, 2)$ such that $x = c_1\beta + \overline{x}_1 = c_2\beta + \overline{x}_2$. Since $\overline{x}_1, \overline{x}_2 \in H$ and $\beta \notin H$, we must have $c_1 = c_2$ and hence $\overline{x}_1 = \overline{x}_2$. It follows $\overline{x}_1 \in \mathrm{Cone}(\overline{R}_1 \cap \overline{R}_2)$ and hence $x \in \mathrm{Cone}(R_1 \cap R_2)$.

(c) Finally, let $\mathrm{rk}(\beta^{\preccurlyeq}) = n$, β not be detachable in $(\beta^{\preccurlyeq})$, and $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ be a bipartition of $(\beta^{\preccurlyeq})$. By definition, each of R_1 and R_2 is contained in exactly one of J_{i} and J_{f}. If both are contained in J_{i}, or both in J_{f}, we are reduced to case (b). Otherwise, we may assume $R_1 \subseteq J_{\mathrm{i}}$, $R_2 \subseteq J_{\mathrm{f}}$, $R_1 \cap (J_{\mathrm{i}} \smallsetminus J_{\mathrm{f}}) \neq \varnothing$, and $R_2 \cap (J_{\mathrm{f}} \smallsetminus J_{\mathrm{i}}) \neq \varnothing$. Let H be a separating hyperplane for the bipartition $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$, and $\overline{R}_i = R_i \cap H$, for $i = 1, 2$. For a fixed i in $\{1, 2\}$, $\mathrm{Conv}(R_i)$ is contained in one of the half-spaces determined by H in E, hence $H \cap \mathrm{Cone}(R_i) = \mathrm{Cone}(\overline{R}_i)$. Moreover, $\mathrm{Cone}(R_1)$ and $\mathrm{Cone}(R_2)$ belong to opposite half-spaces with respect to H, hence, $\mathrm{Cone}(R_1) \cap \mathrm{Cone}(R_2) = \mathrm{Cone}(\overline{R}_1) \cap \mathrm{Cone}(\overline{R}_2)$. Moreover, $R_1 \cap R_2 = \overline{R}_1 \cap \overline{R}_2$. Arguing by induction as in case (b1), we obtain $\mathrm{Cone}(\overline{R}_1) \cap \mathrm{Cone}(\overline{R}_2) \subseteq \mathrm{Cone}(\overline{R}_1 \cap \overline{R}_2)$, hence the claim. □

Corollary 6.5 and Propositions 6.6 and 6.7 imply directly the following theorem, which is Theorem 1.1.

Theorem 6.8. *Let F be a facet of $\mathcal{P}$, I_F be the corresponding facet ideal, and*

$$\mathcal{T}'_{I_F} = \{\mathrm{Conv}(R) \mid R \in \mathcal{R}_{I_F},\ \mathrm{rk}(R) = n\}.$$

Then $\mathcal{T}'_{I_F}$ is a triangulation of F.

Corollary 6.9. *Each reduced subset in I is contained in a maximal reduced subset. Moreover, each maximal reduced subset in I is a linear basis of* E.

Proof. Let R_0 be a reduced subset in I such that $\mathrm{rk}(R_0) < n$. Let $x = \sum_{\beta \in R_0} c_\beta \beta$ with $c_\beta > 0$ for all $\beta \in R_0$. By Corollary 6.5, there exists a reduced subset R in I such that $\mathrm{rk}(R) = n$ and $x \in \mathrm{Conv}(R)$. Then, by Proposition 6.7, $x \in \mathrm{Conv}(R_0 \cap R)$. By assumption, for each proper subset R' of R_0, $x \notin \mathrm{Conv}(R')$, hence $R_0 \cap R = R_0$. It follows $R_0 \subsetneqq R$. Thus, a maximal reduced subset in I has rank n. By Proposition 6.6, it is also linearly independent, hence, it is a linear basis of E. □

We can finally prove the following result, which is equivalent to Theorem 1.2. The proof refers to the case by case analysis of Proposition 5.11.

Theorem 6.10. *Let F be a facet of $\mathcal{P}$, I_F be the corresponding facet ideal, and R be a maximal reduced subset in I_F. Then R is a $\mathbb{Z}$-basis of the sub-lattice of $L(\Phi)$ generated by $(\Pi \smallsetminus \{\alpha_F\}) \cup \{m_{\alpha_F}\alpha_F\}$, where α_F is the simple root such that $F = F_{\alpha_F}$.*

Proof. Let $I = I_F$. By Proposition 3.2 and Remark 3.3, it suffices to prove the claim in case I is an abelian nilradical of Φ^+, i.e. $m_{\alpha_F} = 1$. Thus we have $\alpha_F = \alpha_I$ and $I = (\alpha_I^{\leqslant})$, as before. Under this assumption, we have to prove that R is a $\mathbb{Z}$-basis of $L(\Phi)$.

Let $\preccurlyeq$ be a triangulation order of I and $\beta = \min_{\preccurlyeq} R$. If β is detachable in $(\beta^{\preccurlyeq})$, let $J = (\beta^{\preccurlyeq}) \smallsetminus \{\beta\}$. If β is not detachable in $(\beta^{\preccurlyeq})$, let $\{J_{\mathrm{i}}, J_{\mathrm{f}}\}$ be a bipartition of $(\beta^{\preccurlyeq})$ such that β belongs to J_{i} and J_{f} and is detachable in them. In this case, R is contained in exactly one of J_{i} and J_{f}: we define $J = J_{\mathrm{i}}$ if $R \subseteq J_{\mathrm{i}}$, and $J = J_{\mathrm{f}}$ otherwise. In any case, let H be a detaching hyperplane for β in J. Then, $\mathrm{Red}(\beta) \cap J = H \cap J$, hence $R \smallsetminus \{\beta\}$ is a reduced subset in the abelian nilradical $I \cap H$ of $(\Phi \cap H)^+$. Since $\mathrm{rk}(R \smallsetminus \{\beta\}) = n - 1$, also $\mathrm{rk}(I \cap H) = \mathrm{rk}(\Phi \cap H) = n - 1$. In particular $I \cap H$ has nontrivial intersection with each irreducible component of $\Phi \cap H$. By Lemma 3.4, each of these intersections is a nontrivial abelian nilradical in its irreducible component, hence, by induction on the dimension, $R \smallsetminus \{\beta\}$ is a $\mathbb{Z}$-basis of $L(\Phi \cap H)$.

Now, we first consider the case in which β is long and is equal to $\min J$ or $\max J$ with respect to standard partial order. In this case, as seen in the proof of Lemma 5.7, we may take $H = (\beta^\vee - \check{\omega}_{\alpha_I})^\perp$. It follows directly that all simple roots different from α_I and perpendicular to β belong to H. For all other simple roots $\alpha \neq \alpha_I$, either $(\alpha, \beta^\vee) = 1$ and $\beta - \alpha \in H$, or $(\alpha, \beta^\vee) = -1$ and $\beta + \alpha \in H$. By the induction assumption, we obtain that, for all $\alpha \in \Pi \smallsetminus \{\alpha_I\}$, either α, or one of $\beta \pm \alpha$, is an integral linear combination of $R \smallsetminus \{\beta\}$. It follows that the $\mathbb{Z}$-span of R contains

$(\Pi \setminus \alpha_I)$. Since $\beta \in R$ and $c_{\alpha_I}(\beta) = 1$, it follows that the $\mathbb{Z}$-span of R contains Π, which yields the claim.

Looking at the proof of Proposition 5.11, we can check that also in the remaining cases we may take H so as to satisfy the following condition: for all $\alpha \in \Pi \setminus \{\alpha_I\}$, either $\alpha \in H$, or one of $\beta \pm \alpha \in H$. Arguing as in the previous case, we obtain that R is a $\mathbb{Z}$-basis of $L(\Phi)$. $\square$

7. Concluding remarks

We may transfer a triangulation $\mathcal{T}_F$ of a standard parabolic facet F to all facets in its orbit, by the action of the Weyl group W. If $\overline{w} \in W$ and $\mathrm{Stab}(F) = \{w \in W \mid wF = F\}$, then, for all $v \in \overline{w}\,\mathrm{Stab}(F)$ we have that $v\mathcal{T}_F$ is a triangulation of $\overline{w}F$. Thus, a set of representatives of the left cosets in $W/\mathrm{Stab}(F)$ determines an extension of $\mathcal{T}_F$ to the whole orbit WF. We can prove that, by a suitable choice of the coset representatives for all standard parabolic facets, we may obtain a triangulation of the whole boundary $\partial\mathcal{P}$ of the root polytope $\mathcal{P}$. Let $\mathcal{T}$ be a triangulation of $\partial\mathcal{P}$ obtained by extending, through the action of W, the triangulations of the standard parabolic facets provided in Section 6. If we set $T_0 = \mathrm{Conv}(T \cup \{\underline{0}\})$, for or each $T \in \mathcal{T}$, then $\mathcal{T}_0 := \{T_0 \mid T \in \mathcal{T}\}$ is a triangulation of $\mathcal{P}$. Theorem 6.10 allows to compute the volumes $\mathrm{Vol}(T_0)$, which are constant on each facet orbit. Thus, the explicit enumeration of the maximal reduced subsets of facet ideals, together with the results in [6] on face orbits, would allow to compute the volume of $\mathcal{P}$. For the root types A and C, this is done in [4]. The proof of Proposition 5.11 gives an explicit procedure for enumerating the maximal reduced subsets, hence provides an effective way for making a similar computation for the remaining root types.

In [4] it is also proved that, for the root types A and C, the triangulation $\mathcal{T}_0$ of $\mathcal{P}$ restricts to a triangulation of the positive root polytope $\mathcal{P}^+ = \mathrm{Conv}(\Phi^+ \cup \{\underline{0}\})$. In fact, this is a proof that, for these root types, the intersection of $\mathcal{P}$ with the cone on Φ^+ is equal to $\mathcal{P}^+$. This is one of the special properties of the root polytope that hold only for the types A and C (see also [5]). Indeed, it is easy to see that, for all other root types, $\mathcal{P}^+$ is properly contained in $\mathcal{P} \cap \mathrm{Cone}(\Phi^+)$ [10]. Hence, in these cases, from a triangulation of the standard parabolic facets, we cannot obtain a triangulation of the positive root polytope in a natural way.

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Paola Cellini, Università di Chieti e Pescara, Dipartimento di Ingegneria e Geologia, Viale Pindaro 42, 65127 Pescara PE, Italy
E-mail : `pcellini@unich.it`