



ALGEBRAIC COMBINATORICS


Megumi Harada & Martha E. Precup

The cohomology of abelian Hessenberg varieties and the Stanley–Stembridge conjecture

Volume 2, issue 6 (2019), p. 1059-1108.

http://alco.centre-mersenne.org/item/ALCO_2019__2_6_1059_0

© The journal and the authors, 2019.
Some rights reserved.

 This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>

Access to articles published by the journal *Algebraic Combinatorics* on
the website <http://alco.centre-mersenne.org/> implies agreement with the
Terms of Use (<http://alco.centre-mersenne.org/legal/>).



Algebraic Combinatorics is member of the
Centre Mersenne for Open Scientific Publishing
www.centre-mersenne.org



The cohomology of abelian Hessenberg varieties and the Stanley–Stembridge conjecture

Megumi Harada & Martha E. Precup

ABSTRACT We define a subclass of Hessenberg varieties called abelian Hessenberg varieties, inspired by the theory of abelian ideals in a Lie algebra developed by Kostant and Peterson. We give an inductive formula for the \mathfrak{S}_n -representation on the cohomology of an abelian regular semisimple Hessenberg variety with respect to the action defined by Tymoczko. Our result implies that a graded version of the Stanley–Stembridge conjecture holds in the abelian case, and generalizes results obtained by Shareshian–Wachs and Teff. Our proof uses previous work of Stanley, Gasharov, Shareshian–Wachs, and Brosnan–Chow, as well as results of the second author on the geometry and combinatorics of Hessenberg varieties. As part of our arguments, we obtain inductive formulas for the Poincaré polynomials of regular abelian Hessenberg varieties.

1. INTRODUCTION

Hessenberg varieties in type A are subvarieties of the full flag variety $\mathcal{F}lags(\mathbb{C}^n)$ of nested sequences of linear subspaces in \mathbb{C}^n . These varieties are parameterized by a choice of linear operator $X \in \mathfrak{gl}(n, \mathbb{C})$ and Hessenberg function $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$. We denote the corresponding Hessenberg variety by $\mathcal{H}ess(X, h)$. The geometry and (equivariant) topology of Hessenberg varieties has been studied extensively since the late 1980s [7, 6]. This subject lies at the intersection of, and makes connections between, many research areas such as geometric representation theory, combinatorics, and algebraic geometry and topology.

In this manuscript, we are concerned with the connection between the geometry and topology of Hessenberg varieties and the famous Stanley–Stembridge conjecture in combinatorics, which states that the chromatic symmetric function of the incomparability graph of a so-called $(3+1)$ -free poset is e -positive, i.e. it is a non-negative linear combination of elementary symmetric functions [27, Conjecture 5.5] (see also [25]). Guay-Paquet has subsequently proved that this conjecture, which we refer to below as the “original Stanley–Stembridge conjecture,” can be reduced to the statement that the chromatic symmetric function of the incomparability graph of a unit interval order is e -positive [11], and we refer to the latter statement as the ungraded Stanley–Stembridge conjecture.

Manuscript received 30th July 2018, revised 2nd March 2019, accepted 19th March 2019.

KEYWORDS. Stanley–Stembridge conjecture, symmetric functions, e -positivity, Hessenberg varieties, abelian ideal.

ACKNOWLEDGEMENTS. The first author is partially supported by an NSERC Discovery Grant and a Canada Research Chair (Tier 2) award.

Shareshian and Wachs linked the ungraded Stanley–Stembridge conjecture to Hessenberg varieties via the “dot action” \mathfrak{S}_n -representation on the cohomology ring of a regular semisimple Hessenberg variety defined by Tymoczko [30], as we now explain. The Hessenberg variety $\mathcal{H}ess(\mathbf{S}, h)$ is called a regular semisimple Hessenberg variety if \mathbf{S} is a regular semisimple element of $\mathfrak{gl}(n, \mathbb{C})$. Shareshian and Wachs established a bijection between Hessenberg functions and unit interval orders [22, Proposition 4.1]; their bijection associates each Hessenberg function h to the incomparability graph of a unit interval order, here denoted by Γ_h . In addition, Shareshian and Wachs defined the chromatic quasisymmetric function $X_\Gamma(\underline{x}, t)$ of a graph Γ , which refines Stanley’s chromatic symmetric function in the sense that $X_\Gamma(\underline{x}, 1)$ is Stanley’s chromatic symmetric function. They then formulated a conjecture relating the chromatic quasisymmetric function of the graph Γ_h to the image of the character of the dot action representation on $H^*(\mathcal{H}ess(\mathbf{S}, h))$ under the characteristic map. This conjecture, known as the Shareshian–Wachs conjecture, provides the link between Hessenberg varieties and chromatic symmetric (and quasi-symmetric) functions. We discuss it in greater detail in Section 2.2 below.

Since cohomology rings are naturally graded by degree, the Shareshian–Wachs conjecture actually suggests that one should consider a graded version of the Stanley–Stembridge conjecture. Specifically, the “graded Stanley–Stembridge conjecture” (see [22, Conjecture 10.4]) states that the coefficient of t^i in the chromatic quasisymmetric function $X_{\Gamma_h}(\underline{x}, t)$ is e -positive. We discuss this refined version of the ungraded Stanley–Stembridge conjecture in Section 2.2 and state it formally in Conjecture 2.8. To emphasize, the statement of the Shareshian–Wachs conjecture provides the necessary link between the cohomology of Hessenberg varieties and the graded Stanley–Stembridge conjecture, thus yielding a new way of attacking both the ungraded and the graded versions of the conjecture.

The Shareshian–Wachs conjecture was proved in 2015 by Brosnan and Chow [3] (also independently by Guay-Paquet [12]) by showing a remarkable relationship between the Betti numbers of different Hessenberg varieties. (Direct computations of cohomology rings of certain Hessenberg varieties also yield partial proofs of the Shareshian–Wachs conjecture; see [1, 2].) As we have explained above, it then follows that in order to prove the graded Stanley–Stembridge conjecture, it suffices to prove that the cohomology $H^{2i}(\mathcal{H}ess(\mathbf{S}, h))$ for each i is a non-negative combination of the tabloid representations M^λ [8, Part II, Section 7.2] of \mathfrak{S}_n for λ a partition of n . In other words, given the decomposition

$$(1) \quad H^{2i}(\mathcal{H}ess(\mathbf{S}, h)) = \sum_{\lambda \vdash n} c_{\lambda, i} M^\lambda$$

in the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ of \mathfrak{S}_n , it suffices to show that the coefficients $c_{\lambda, i}$ are non-negative.

The above discussion explains the motivation for this manuscript, and we now describe our main results. Let $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function. Our approach to the graded Stanley–Stembridge conjecture is by induction. Roughly, the idea is as follows. From any Hessenberg function h we can construct the corresponding incomparability graph Γ_h (made precise in Section 2.3). Previous results of Stanley show that the acyclic orientations of Γ_h , and their corresponding sets of sinks, encode information about the coefficients $c_{\lambda, i}$. We develop this idea further by decomposing the set of acyclic orientations according to their sink sets, and make a key observation (Proposition 4.10) that, if the size of a sink set is maximal, then the set of acyclic orientations with that fixed sink set corresponds precisely to the set of

all acyclic orientations on a smaller incomparability graph. This observation sets the stage for an inductive argument.

Any Hessenberg function corresponds uniquely to a certain subset I_h of the negative roots of $\mathfrak{gl}(n, \mathbb{C})$. In this manuscript, in the special case when I_h is abelian (cf. Definition 3.1 below), we are able to fully implement an argument yielding an inductive formula for the coefficients of the tabloid representations. A rough statement of our main result is as follows (definitions are in Section 2.1, Section 2.2 and Section 4); the precise statement is Theorem 6.1. The idea is that the coefficients $c_{\lambda, i}$ above, associated to a Hessenberg variety in $\mathcal{F}lags(\mathbb{C}^n)$ for $n \geq 3$, can be computed using the coefficients associated to certain Hessenberg varieties in the flag variety $\mathcal{F}lags(\mathbb{C}^{n-2})$.

THEOREM 1.1. *Let $n \geq 3$ be a positive integer and $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a Hessenberg function such that the ideal I_h is abelian. Let S denote a regular semisimple element in the Lie algebra of $\mathfrak{gl}(n, \mathbb{C})$. Let $i \geq 0$ be a non-negative integer. Regard the cohomology $H^{2i}(\mathcal{H}ess(S, h))$ as a \mathfrak{S}_n -representation using Tymoczko’s dot action. Then, in the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ we have the equality*

$$(2) \quad H^{2i}(\mathcal{H}ess(S, h)) = c_{(n), i} M^{(n)} + \sum_{T \in SK_2(\Gamma_h)} \left(\sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \deg(T)}^T M^{(\mu_1 + 1, \mu_2 + 1)} \right)$$

where the set $SK_2(\Gamma_h)$ is a certain collection of subsets of the vertices of Γ_h and the coefficients $c_{\mu, i - \deg(T)}^T$ are the coefficients as in (1) associated to a Hessenberg function $h_T : \{1, 2, \dots, n - 2\} \rightarrow \{1, 2, \dots, n - 2\}$ for a Hessenberg variety in $\mathcal{F}lags(\mathbb{C}^{n-2})$.

The technical details of the induction argument leading to Theorem 1.1 require the use, among other things, of Brosnan and Chow’s proof of the Shareshian–Wachs conjecture, as well as the second author’s combinatorial characterization of the Betti numbers of regular Hessenberg varieties. In fact, the technical core of the paper consists of two inductive formulas for the Poincaré polynomials of regular Hessenberg varieties in the abelian case. These formulas are stated in Proposition 6.5 and Proposition 6.6 and are of independent interest.

It is quite straightforward to prove the graded Stanley–Stembridge conjecture for the abelian case based on our inductive formula in Theorem 1.1, and we record this argument in Corollary 7.26. Our result generalizes previous results. Indeed, in the case when h satisfies $h(3) = \dots = h(n) = n$, Shareshian and Wachs obtained results on the corresponding chromatic quasisymmetric function which, given Brosnan and Chow’s proof of the Shareshian–Wachs conjecture, implies Corollary 7.26 for that case. Separately, Teff [28, Theorem 4.20] proved the case when h corresponds to a maximal standard parabolic Lie subalgebra \mathfrak{p} of $\mathfrak{gl}(n, \mathbb{C})$. Both instances are special cases of our result, as we explain in Section 3. Separately, we also note that Gebhard and Sagan have proved the original Stanley–Stembridge conjecture for a collection of graphs called K_α -chains [10, Corollary 7.7]. Their result does not subsume, nor is it subsumed by, the case considered in this manuscript, but it is of independent interest. Since the first version of this manuscript appeared on the arXiv, Cho and Huh have posted another independent proof of the graded Stanley–Stembridge conjecture in the same case we consider below [5].

As part of our arguments, we define the height of an ideal of negative roots using the lower central series of an ideal in a Lie algebra. An ideal is abelian precisely when the height is either 1 or 0, so we can interpret Theorem 1.1 as a “base case” for an argument for the graded Stanley–Stembridge conjecture using induction on the height of the ideal I_h . We intend to explore this further in future work.

As already mentioned, the graded Stanley–Stembridge conjecture implies the ungraded Stanley–Stembridge conjecture simply by summing over all i , or, in the language of chromatic quasisymmetric functions, by “setting t equal to 1”. We record this fact in Proposition 2.11. We note here that the “abelian case” considered in Theorem 6.1 (and Corollary 7.26) corresponds, in combinatorial language, to the case in which the vertices of the graph Γ_h can be partitioned into two disjoint cliques. The fact that the coefficients $c_\lambda = \sum_{i \geq 0} c_{\lambda, i}$ are non-negative in this case was originally stated by Stanley in [25, Corollary 3.6] as a corollary to [25, Theorem 3.4]; moreover, this fact is also equivalent to [27, Remark 4.4]. However, [25, Theorem 3.4] is incorrect as stated [24], and the equivalence of [27, Remark 4.4] and [25, Corollary 3.6] is not explicit in [27, 25]. Thus, our Corollary 7.26 (together with Proposition 2.11) records a new and explicit proof of this fact.

We now give a brief overview of the contents of the paper. Section 2 is devoted to background material. Specifically, Section 2.1 is a crash course on Hessenberg varieties. Section 2.2 establishes the terminology for discussing the \mathfrak{S}_n -representations $H^{2i}(\mathcal{Hess}(\mathcal{S}, h))$, and gives a more detailed account of the relation between the Stanley–Stembridge conjecture and our results. Section 2.3 recalls the language of incomparability graphs in the setting of Hessenberg functions and states a result of Stanley connecting acyclic orientations on this graph to the \mathfrak{S}_n -representations above. Section 2.4 recounts Gasharov’s definition of a P_h -tableau and a result relating these P_h -tableaux to the same \mathfrak{S}_n -representations above. We then begin our work in earnest in Section 3 where we define abelian Hessenberg varieties and briefly discuss the relation between this notion and the cases of Hessenberg varieties previously studied in the literature. In Section 4 we focus attention on the sink sets of an acyclic orientation of an incomparability graph, and introduce the notion of sink-set size. In Section 5 we link the subjects of Sections 3 and 4 using a new invariant of an ideal called the height. Sections 6 and 7 form the technical core of the paper, where we state and prove our main results. Finally, Section 8 states a conjecture which, if true, would represent a first step towards generalizing the techniques in this paper to prove the full Stanley–Stembridge conjecture for all possible heights.

2. THE SETUP AND BACKGROUND

Let n be a positive integer. We denote by $[n]$ the set of positive integers $\{1, 2, \dots, n\}$. We work in type A throughout, so $\mathrm{GL}(n, \mathbb{C})$ is the group of invertible $n \times n$ complex matrices and $\mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ consisting of all $n \times n$ complex matrices.

2.1. HESSENBERG VARIETIES. Hessenberg varieties in Lie type A are subvarieties of the (full) flag variety $\mathcal{F}lags(\mathbb{C}^n)$, which is the collection of sequences of nested linear subspaces of \mathbb{C}^n :

$$\mathcal{F}lags(\mathbb{C}^n) := \{V_\bullet = (\{0\} \subset V_1 \subset \dots \subset V_{n-1} \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\}.$$

A Hessenberg variety in $\mathcal{F}lags(\mathbb{C}^n)$ is specified by two pieces of data: a Hessenberg function and a choice of an element in $\mathfrak{gl}(n, \mathbb{C})$. We have the following.

DEFINITION 2.1. *A Hessenberg function is a function $h : [n] \rightarrow [n]$ such that $h(i) \geq i$ for all $i \in [n]$ and $h(i+1) \geq h(i)$ for all $i \in [n-1]$. We frequently write a Hessenberg function by listing its values in sequence, i.e. $h = (h(1), h(2), \dots, h(n))$.*

We now introduce some terminology associated to a given Hessenberg function.

DEFINITION 2.2. Let $h : [n] \rightarrow [n]$ be a Hessenberg function. The associated Hessenberg space is the linear subspace H of $\mathfrak{gl}(n, \mathbb{C})$ specified as follows:

$$(3) \quad \begin{aligned} H &:= \{A = (a_{ij})_{i,j \in [n]} \in \mathfrak{gl}(n, \mathbb{C}) \mid a_{ij} = 0 \text{ if } i > h(j)\} \\ &= \text{span}_{\mathbb{C}}\{E_{ij} \mid i, j \in [n] \text{ and } i \leq h(j)\} \end{aligned}$$

where $E_{i,j}$ is the usual elementary matrix with a 1 in the (i, j) -th entry and 0's elsewhere.

It is important to note that H is frequently *not* a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. However, it *is* stable under the conjugation action of the usual maximal torus T (of invertible diagonal matrices) in $GL(n, \mathbb{C})$, and the E_{ij} appearing in (3) are exactly the T -eigenvectors. It is also straightforward to see that

$$(4) \quad [\mathfrak{b}, H] \subseteq H$$

where $[\cdot, \cdot]$ denotes the usual Lie bracket in $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{b} = \text{Lie}(B)$ is the Lie algebra of the Borel subgroup B of upper-triangular matrices in $GL(n, \mathbb{C})$.

Let $\mathfrak{h} \subseteq \mathfrak{gl}(n, \mathbb{C})$ denote the Cartan subalgebra of diagonal matrices, and let t_i denote the coordinate on \mathfrak{h} reading off the (i, i) -th matrix entry along the diagonal. Denote the root system of $\mathfrak{gl}(n, \mathbb{C})$ by Φ . Then the positive roots Φ^+ of $\mathfrak{gl}(n, \mathbb{C})$ are $\Phi^+ = \{t_i - t_j \mid 1 \leq i < j \leq n\}$ where $\gamma = t_i - t_j \in \Phi^+$ corresponds to the root space spanned by E_{ij} , denoted \mathfrak{g}_γ . Similarly, the negative roots Φ^- of $\mathfrak{gl}(n, \mathbb{C})$ are $\Phi^- = \{t_i - t_j \mid 1 \leq j < i \leq n\}$. We denote the simple positive roots in Φ^+ by $\Delta = \{\alpha_i := t_i - t_{i+1} \mid 1 \leq i \leq n - 1\}$.

Note that the pairs (i, j) with $i > j$ and $i \leq h(j)$ correspond precisely to those negative roots $\gamma \in \Phi^-$ whose associated root spaces \mathfrak{g}_γ are contained in H . Motivated by this, we fix the following notation:

$$\Phi_h^- := \{t_i - t_j \in \Phi^- \mid E_{ij} \in H\} = \{t_i - t_j \mid i > j \text{ and } i \leq h(j)\}$$

and

$$\Phi_h := \Phi_h^- \sqcup \Phi^+ = \{t_i - t_j \in \Phi \mid i \leq h(j)\}.$$

It is clear that h is uniquely determined by either Φ_h^- or Φ_h .

Recall that an *ideal* (also called an *upper-order ideal*) I of Φ^- is defined to be a collection of (negative) roots such that if $\alpha \in I$, $\beta \in \Phi^-$, and $\alpha + \beta \in \Phi^-$, then $\alpha + \beta \in I$. The relation (4) immediately implies that

$$I_h := \Phi^- \setminus \Phi_h^-$$

is an ideal in Φ^- . We call it the *ideal corresponding to h* . In fact, the association taking a Hessenberg function to its corresponding ideal I_h defines a bijection from the set of Hessenberg functions to ideals in Φ^- , as noted by Sommers and Tymoczko in [23, Section 10].

It is conceptually useful to express the sets Φ_h , Φ_h^- , and I_h pictorially. We illustrate this by an example.

EXAMPLE 2.3. Let $n = 6$. Figure 1 contains the pictures corresponding to the Hessenberg function $h = (3, 4, 5, 6, 6, 6)$. The leftmost square grid contains a star in the (i, j) -th box exactly if the (i, j) -th matrix entry is allowed to be non-zero for $A \in H$, or equivalently, either $i = j$, or, the corresponding root $t_i - t_j$ of $\mathfrak{gl}(n, \mathbb{C})$ is contained in Φ_h . The center square grid contains a star in the (i, j) -th box precisely if the corresponding root is contained in Φ_h^- . Finally, the rightmost grid contains a star in the (i, j) -th box if and only if the corresponding root is contained in I_h , i.e. it is the complement of Φ_h . This illustrates why some authors refer to I_h as (the roots corresponding to) the “opposite Hessenberg space”.

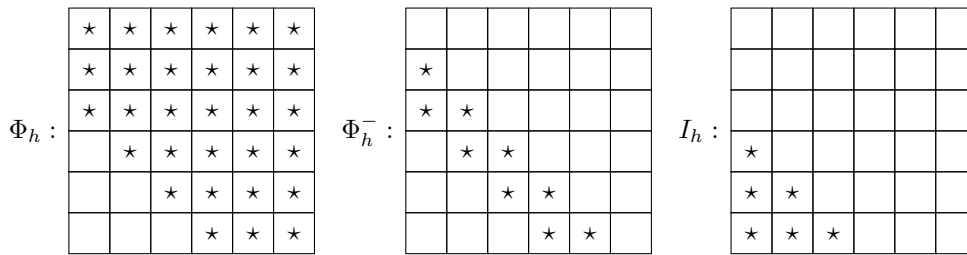


FIGURE 1. The pictures of Φ_h , Φ_h^- , and I_h for $h = (3, 4, 5, 6, 6, 6)$.

Let $h : [n] \rightarrow [n]$ be a Hessenberg function and X be an $n \times n$ matrix in $\mathfrak{gl}(n, \mathbb{C})$, which we also consider as a linear operator $\mathbb{C}^n \rightarrow \mathbb{C}^n$. Then the *Hessenberg variety* $\mathcal{Hess}(X, h)$ associated to h and X is defined to be

$$(5) \quad \mathcal{Hess}(X, h) := \{V_\bullet \in \mathcal{Flags}(\mathbb{C}^n) \mid XV_i \subset V_{h(i)} \text{ for all } i \in [n]\} \subset \mathcal{Flags}(\mathbb{C}^n).$$

In this paper we focus on certain special cases of Hessenberg varieties. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a composition of n in the sense that $\lambda_1 + \lambda_2 + \dots + \lambda_n = n$ and $\lambda_i \geq 0$ for all i . A linear operator is *regular of Jordan type* λ if its standard Jordan canonical form has block sizes given by λ_1, λ_2 , etc., and no two distinct blocks have the same eigenvalue. Note that if $g \in \text{GL}(n, \mathbb{C})$, then $\mathcal{Hess}(X, h)$ and $\mathcal{Hess}(gXg^{-1}, h)$ can be identified via the action of $\text{GL}(n, \mathbb{C})$ on $\mathcal{Flags}(\mathbb{C}^n)$ [29]. For concreteness in what follows, for a given λ as above we set the notation

X_λ is a (fixed) regular matrix in Jordan canonical form of Jordan type λ

and we refer to the corresponding Hessenberg variety $\mathcal{Hess}(X_\lambda, h)$ as a *regular Hessenberg variety*.

Two special cases are of particular interest. Namely, if $\lambda = (n, 0, \dots, 0) = (n)$, then we may take the corresponding regular operator to be the regular nilpotent operator which we denote by N , i.e. N is the matrix whose Jordan form consists of exactly one Jordan block with corresponding eigenvalue equal to 0. The regular Hessenberg variety $\mathcal{Hess}(N, h)$ is called a *regular nilpotent Hessenberg variety*. Similarly let S denote a regular semisimple matrix in $\mathfrak{gl}(n, \mathbb{C})$, i.e. a matrix which is diagonalizable with distinct eigenvalues. This corresponds to the other extreme case, namely, $\lambda = (1, 1, 1, \dots, 1)$. We call $\mathcal{Hess}(S, h)$ a *regular semisimple Hessenberg variety*.

2.2. THE STANLEY–STEMBRIDGE CONJECTURE IN TERMS OF TYMOCZKO’S DOT ACTION REPRESENTATION. As already discussed in the Introduction, the main motivation of this manuscript is to study a graded version of the Stanley–Stembridge conjecture (Conjecture 2.8 below), stated in terms of the \mathfrak{S}_n -representation on the cohomology rings of regular semisimple Hessenberg varieties defined by Tymoczko [30]. Tymoczko’s *dot action* preserves the grading on these cohomology rings (which is concentrated in even degrees). The structure of this section is as follows. We first review basic facts and establish notation for partitions and \mathfrak{S}_n -representations. We then give in Conjecture 2.8 a precise statement of the graded Stanley–Stembridge conjecture. We also state the ungraded Stanley–Stembridge conjecture, and briefly recount how a solution to Conjecture 2.8 implies the ungraded Stanley–Stembridge conjecture (cf. discussion in [3, 22]). The rest of the section is a brief review of standard representation theory facts and a statement of a fundamental result of Brosnan and Chow.

A *partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 + \lambda_2 + \dots + \lambda_n = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. If λ is a partition of n we write $\lambda \vdash n$. We say a partition $\lambda \vdash n$ has k parts and write $\text{parts}(\lambda) = k$ if $\lambda_k \neq 0$ and $\lambda_{k+1} = \dots = \lambda_n = 0$. (Alternatively, λ has k parts if and only if the Young diagram corresponding to λ has precisely k rows.) For simplicity if $\text{parts}(\lambda) = k$ then we write $\lambda = (\lambda_1, \dots, \lambda_k)$ instead of $\lambda = (\lambda_1, \dots, \lambda_k, 0, 0, \dots, 0)$. Moreover, for $\nu \vdash n$ a partition of n , we let $\mathfrak{S}_\nu \subseteq \mathfrak{S}_n$ denote the *Young subgroup* of \mathfrak{S}_n corresponding to ν . Concretely, if $\nu = (\nu_1, \nu_2, \dots, \nu_k)$ has k parts then \mathfrak{S}_ν is the subgroup

$$\mathfrak{S}_{1, \dots, \nu_1} \times \mathfrak{S}_{\nu_1+1, \dots, \nu_1+\nu_2} \times \dots \times \mathfrak{S}_{(\sum_{\ell=1}^{k-1} \nu_\ell)+1, \dots, n} \subseteq \mathfrak{S}_n$$

where $\mathfrak{S}_{i, i+1, \dots, j}$ denotes the permutations of the set $\{i, i+1, \dots, j\}$ for each $1 \leq i < j \leq n$.

Following Fulton [8, Part II, Section 7.2], we denote by M^λ the complex vector space with basis the set of tabloids $\{T\}$ of shape λ , where λ is a partition of n . Since \mathfrak{S}_n acts on the set of tabloids, M^λ is a \mathfrak{S}_n -representation. Our main theorem concerns the decomposition of $H^{2i}(\text{Hess}(\mathcal{S}, h))$ into M^λ 's, but to study this, we first decompose $H^{2i}(\text{Hess}(\mathcal{S}, h))$ into irreducible representations. Denote by \mathcal{S}^λ the Specht module corresponding to λ . It is well-known that each \mathcal{S}^λ is irreducible and that any irreducible \mathfrak{S}_n -representation is isomorphic to \mathcal{S}^λ for some λ [8, Section 7.2, Proposition 1]. Thus we conclude that there exist non-negative integers d_λ and $d_{\lambda,i}$ such that

$$(6) \quad H^*(\text{Hess}(\mathcal{S}, h)) \cong \bigoplus_{\lambda \vdash n} d_\lambda \mathcal{S}^\lambda \quad \text{and} \quad H^{2i}(\text{Hess}(\mathcal{S}, h)) \cong \bigoplus_{\lambda \vdash n} d_{\lambda,i} \mathcal{S}^\lambda$$

as \mathfrak{S}_n -representations. Note $d_\lambda = \sum_{i \geq 0} d_{\lambda,i}$ for any λ .

There is a well-known formula, called Young's rule, for the decomposition of M^λ into Specht modules. We need some terminology. Recall that the *dominance order* on partitions [8, page 26] is defined as

$$\lambda \preceq \nu \text{ if and only if } \lambda_1 + \dots + \lambda_i \leq \nu_1 + \nu_2 + \dots + \nu_i \text{ for all } i.$$

The following lemmas are straightforward. Let \leq_{lex} denote the usual lexicographic order on \mathbb{Z}^n .

LEMMA 2.4. *Let λ, ν be partitions of n . If $\lambda \preceq \nu$ then $\lambda \leq_{\text{lex}} \nu$.*

LEMMA 2.5. *Let λ, ν be partitions of n . If $\lambda \preceq \nu$ then $\text{parts}(\nu) \leq \text{parts}(\lambda)$.*

We define the following total order on partitions:

$$(7) \quad \lambda \preceq \nu \Leftrightarrow_{\text{def}} (\text{parts}(\nu) < \text{parts}(\lambda)) \text{ or } (\text{parts}(\nu) = \text{parts}(\lambda) \text{ and } \lambda \leq_{\text{lex}} \nu).$$

In words, this total order first compares partitions based on the number of parts, and then breaks ties using the usual lex order. The following is immediate from the above two lemmas.

LEMMA 2.6. *The order \preceq is a refinement of the dominance order, i.e. for $\lambda, \nu \vdash n$, if $\lambda \preceq \nu$ then $\lambda \preceq \nu$.*

For a pair of partitions λ and ν of n , the *Kostka number* $K_{\nu\lambda}$ is defined [8, Part I, Section 2.2] to be the number of semistandard Young tableaux of shape ν and weight/content λ .

FACT 2.7 ([4, Lemma 3.7.3]). *For λ, ν partitions of n , we have $K_{\nu\lambda} \neq 0$ if and only if $\lambda \preceq \nu$. Moreover, $K_{\lambda\lambda} = 1$ for all partitions $\lambda \vdash n$.*

Let $K := (K_{\nu\lambda})$ denote the *Kostka matrix* with entries the Kostka numbers with partitions listed in decreasing order with respect to \preceq . By Lemma 2.6 and Fact 2.7 above, K is upper-triangular and has 1's along the diagonal, and in particular, is invertible over \mathbb{Z} . Young's rule [8, Section 7.3, Corollary 1] states that

$$(8) \quad M^\lambda \cong \mathcal{S}^\lambda \oplus \left(\bigoplus_{\nu \triangleright \lambda} (\mathcal{S}^\nu)^{\oplus K_{\nu\lambda}} \right)$$

as \mathfrak{S}_n -representations. Since the Kostka matrix is invertible over \mathbb{Z} , (8) implies the $\{M^\lambda\}$ form a \mathbb{Z} -basis for the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ of \mathfrak{S}_n . Therefore there exist unique integers c_λ and $c_{\lambda,i}$ such that

$$(9) \quad H^*(\mathcal{H}ess(\mathcal{S}, h)) = \sum_{\lambda \vdash n} c_\lambda M^\lambda \quad \text{and} \quad H^{2i}(\mathcal{H}ess(\mathcal{S}, h)) = \sum_{\lambda \vdash n} c_{\lambda,i} M^\lambda$$

as elements in $\mathcal{R}ep(\mathfrak{S}_n)$. Note that, a priori, the coefficients c_λ and $c_{\lambda,i}$ may be negative. We also have $c_\lambda = \sum_{i \geq 0} c_{\lambda,i}$ for all $\lambda \vdash n$. We can now formulate the *graded Stanley–Stembridge conjecture* which motivates this manuscript; the terminology will be justified below.

CONJECTURE 2.8. *Let n be a positive integer and $h : [n] \rightarrow [n]$ be a Hessenberg function. Then the integers $c_{\lambda,i}$ appearing in (9) are non-negative.*

The main theorem of this manuscript (Theorem 6.1) allows us to deduce the above conjecture in the special case that h is abelian (cf. Definition 3.1). Before proceeding we take a moment to explain how a proof of Conjecture 2.8 implies the ungraded Stanley–Stembridge conjecture. Since this story has been recorded elsewhere (e.g. [3, 22]) we will be brief. We begin with a statement of the ungraded Stanley–Stembridge conjecture. An incomparability graph of a unit interval order is a finite graph $\Gamma = (V, E)$ whose vertices are (distinct) closed unit intervals on the real line, with a single edge joining unit intervals with non-empty intersection. For any finite graph $\Gamma = (V, E)$, a *coloring* of Γ is a function $\kappa : V \rightarrow \{1, 2, 3, \dots\}$ assigning the “color” $\kappa(v)$ to each $v \in V$ and κ is proper if for every edge $e = \{u, v\}$ in E , $\kappa(u) \neq \kappa(v)$. The *chromatic symmetric function* $X_\Gamma(x_1, x_2, \dots)$ is defined as

$$X_\Gamma(\underline{x}) = X_\Gamma(x_1, x_2, \dots) = \sum_{\text{proper } \kappa: V \rightarrow \{1, 2, \dots\}} x^\kappa$$

where $x^\kappa := \prod_{v \in V} x_{\kappa(v)}$. It is not hard to see that X_Γ is symmetric in the variables $\{x_i\}$. A symmetric function is said to be *e-positive* if it can be expressed as a non-negative linear combination of the elementary symmetric functions e_λ . The following is the ungraded Stanley–Stembridge conjecture, which is related to many other deep conjectures, e.g. about immanants.

CONJECTURE 2.9. (*Stanley–Stembridge conjecture*) *Let $\Gamma = (V, E)$ be the incomparability graph of a unit interval order. Then $X_\Gamma(\underline{x})$ is e-positive.*

(In fact, the Stanley–Stembridge conjecture is stated more generally, but Guay-Paquet showed in [11] that the above special case implies the general version.) Shareshian and Wachs linked the Stanley–Stembridge conjecture in [22] to the theory of Hessenberg varieties as follows. For the discussion below, we assume the vertex set V of Γ is a finite subset of $\{1, 2, \dots\}$. Shareshian and Wachs consider a refinement of Stanley's chromatic symmetric polynomial by defining

$$X_\Gamma(\underline{x}, t) := \sum_{\text{proper } \kappa: V \rightarrow \{1, 2, 3, \dots\}} t^{\text{asc } \kappa} x^\kappa$$

where

$$\text{asc}(\kappa) := |\{e = \{i, j\} \in E \mid i < j \text{ and } \kappa(i) < \kappa(j)\}|.$$

This polynomial is called the chromatic quasisymmetric function. Evidently, evaluating $X_\Gamma(\underline{x}, t)$ at $t = 1$ recovers Stanley’s $X_\Gamma(\underline{x})$. Shareshian and Wachs further conjectured that the coefficients of the t^i in $X_\Gamma(\underline{x}, t)$ are related to Hessenberg varieties as follows. Specifically, they show [22, Proposition 4.1] that any incomparability graph Γ of a unit interval order arises as the incomparability graph of an appropriately chosen Hessenberg function $h : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$. Let $\text{ch} : \mathcal{R}ep(\mathfrak{S}_n) \rightarrow \Lambda_{\mathbb{Q}} := \Lambda \otimes \mathbb{Q}$ denote the characteristic map from the representation ring of \mathfrak{S}_n to the ring of symmetric functions in the variables $\underline{x} = (x_1, x_2, \dots)$ with rational coefficients. It is well-known that $\text{ch}(M^\lambda) = h_\lambda$ and $\text{ch}(S^\lambda) = s_\lambda$, where the h_λ (respectively s_λ) are the complete symmetric (respectively Schur) functions. Also let $\omega : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ denote the standard (“Frobenius”) involution on the space of symmetric functions which takes h_λ to the elementary symmetric function e_λ and vice versa. We now state the conjecture of Shareshian and Wachs [22] which is now a theorem thanks to the work of Brosnan and Chow [3].

THEOREM 2.10 ([3, Theorem 129]). *Let Γ be the incomparability graph of a unit interval order and h be the corresponding Hessenberg function. Then the coefficient of t^i in the chromatic quasisymmetric function $X_\Gamma(\underline{x}, t)$ is $\omega(\text{ch}(H^{2i}(\mathcal{H}ess(\mathcal{S}, h))))$.*

We can now make precise the argument that deduces the classical Stanley–Stembridge conjecture from the graded Stanley–Stembridge conjecture (Conjecture 2.8).

PROPOSITION 2.11. *Let Γ and h be as above. If $H^{2i}(\mathcal{H}ess(\mathcal{S}, h)) = \sum_{\lambda \vdash n} c_{\lambda, i} M^\lambda \in \mathcal{R}ep(\mathfrak{S}_n)$ and the $c_{\lambda, i}$ are non-negative for all $\lambda \vdash n$, then $X_\Gamma(\underline{x})$ is e -positive.*

Proof. By construction of the map ch , we have $\text{ch}(H^{2i}(\mathcal{H}ess(\mathcal{S}, h))) = \text{ch}(\sum c_{\lambda, i} M^\lambda) = \sum c_{\lambda, i} \text{ch}(M^\lambda) = \sum c_{\lambda, i} h_\lambda$ is a non-negative linear combination of the h_λ . Thus $\omega(\text{ch}(H^{2i}(\mathcal{H}ess(\mathcal{S}, h)))) = \sum c_{\lambda, i} \omega(h_\lambda) = \sum c_{\lambda, i} e_\lambda$ is a non-negative combination of the e_λ . By Theorem 2.10 the coefficient of t^i in the chromatic quasi-symmetric polynomial $X_\Gamma(\underline{x}, t)$ is e -positive; by evaluation at $t = 1$, the same is true for $X_\Gamma(\underline{x})$. \square

The remainder of this section will be a review of some standard facts in the representation theory of \mathfrak{S}_n as well as a fundamental result of Brosnan and Chow on Hessenberg varieties.

The following lemma is straightforward and probably well-known. Let $V = \bigoplus_{\ell \geq 0} V_\ell$ be a graded \mathfrak{S}_n -representation, i.e. \mathfrak{S}_n preserves each subspace V_ℓ . Let $d_{\lambda, V}$ and $c_{\lambda, V}$ (respectively d_{λ, V_ℓ} , c_{λ, V_ℓ}) denote the integers associated to V (respectively V_ℓ for each $\ell \geq 0$) given by the decomposition of V (respectively V_ℓ) into S^λ ’s and M^λ ’s as elements of $\mathcal{R}ep(\mathfrak{S}_n)$.

LEMMA 2.12. *Let k be a positive integer. Suppose $d_{\lambda, V} = 0$ for all $\lambda \vdash n$ with more than k parts. Then*

- (1) $d_{\lambda, V_\ell} = 0$ for all $\lambda \vdash n$ with more than k parts and for all $\ell \geq 0$,
- (2) $c_{\lambda, V} = 0$ for all $\lambda \vdash n$ with more than k parts, and
- (3) $c_{\lambda, V_\ell} = 0$ for all $\lambda \vdash n$ with more than k parts and for all $\ell \geq 0$.

Proof. Since $d_{\lambda, V} = \sum_\ell d_{\lambda, V_\ell}$, if $d_{\lambda, V} = 0$ then $d_{\lambda, V_\ell} = 0$ for all ℓ . This proves (1). To prove (2), recall first that Young’s rule implies $\mathbf{d}_V = K \mathbf{c}_V$ where \mathbf{d}_V and \mathbf{c}_V are vectors with entries $d_{\lambda, V}$ and $c_{\lambda, V}$ respectively, ordered in such a way that the indexing partitions decrease with respect to \preceq . We have already seen that K is an upper-triangular matrix with 1s along the diagonal. Hence K^{-1} has the same properties.

Also, since the total order \preceq orders partitions by the number of parts, the given hypothesis on the $d_{\lambda,V}$'s implies that the vector \mathbf{d}_V has coordinates all equal to 0 below a certain point. Since $\mathbf{c}_V = K^{-1}\mathbf{d}_V$, we conclude that \mathbf{c}_V must have the same property. The last claim follows from (1) by an argument identical to the discussion above, applied to V_ℓ instead of V . \square

Next we recall some facts which we later use to show that two representations are isomorphic. Given any finite-dimensional representation V of \mathfrak{S}_n and any partition ν of n , we may consider $V^{\mathfrak{S}_\nu}$, the \mathfrak{S}_ν -stable subspace of V . The following is well-known (see e.g. [3, Proposition 10]).

PROPOSITION 2.13. *Let V and W be finite-dimensional representations of \mathfrak{S}_n . Then V and W are isomorphic as \mathfrak{S}_n -representations if and only if*

$$\dim V^{\mathfrak{S}_\nu} = \dim W^{\mathfrak{S}_\nu} \text{ for all partitions } \nu \vdash n.$$

In the setting of the representation ring, there is a similar statement. For two partitions λ, ν of n , we set the notation

$$(10) \quad N_{\lambda,\nu} := \dim(M^\lambda)^{\mathfrak{S}_\nu}.$$

Let $N = (N_{\lambda,\nu})$ denote the matrix with these entries. From [26] we know that

$$N_{\lambda,\nu} = \sum_{\mu \vdash n} K_{\mu,\lambda} K_{\mu,\nu} \quad \text{and hence} \quad N = K^T K$$

where $K = (K_{\lambda\mu})$ is the Kostka matrix and K^T denotes the transpose of K . In particular, since K is invertible over \mathbb{Z} , it follows that N is invertible over \mathbb{Z} . Now suppose we have two elements $\sum a_\lambda M^\lambda, \sum b_\lambda M^\lambda$ in $\mathcal{R}ep(\mathfrak{S}_n)$ where $a_\lambda, b_\lambda \in \mathbb{Z}$ for all $\lambda \vdash n$. By definition, $\sum a_\lambda M^\lambda = \sum b_\lambda M^\lambda$ if and only if $a_\lambda = b_\lambda$ for all $\lambda \vdash n$, or equivalently, $\mathbf{a} = \mathbf{b}$ where $\mathbf{a} = (a_\lambda)_{\lambda \vdash n}$ and $\mathbf{b} = (b_\lambda)_{\lambda \vdash n}$ are (column) vectors with entries a_λ, b_λ respectively. Since N is invertible, the following analogue of Proposition 2.13 for $\mathcal{R}ep(\mathfrak{S}_n)$ is immediate.

PROPOSITION 2.14. *Let $\sum_{\lambda \vdash n} a_\lambda M^\lambda, \sum_{\lambda \vdash n} b_\lambda M^\lambda$ be elements in $\mathcal{R}ep(\mathfrak{S}_n)$. The following are equivalent:*

- (1) $\sum_{\lambda \vdash n} a_\lambda M^\lambda = \sum_{\lambda \vdash n} b_\lambda M^\lambda$ in $\mathcal{R}ep(\mathfrak{S}_n)$,
- (2) $N\mathbf{a} = N\mathbf{b}$, and
- (3) $\sum_{\lambda \vdash n} a_\lambda \dim(M^\lambda)^{\mathfrak{S}_\nu} = \sum_{\lambda \vdash n} b_\lambda \dim(M^\lambda)^{\mathfrak{S}_\nu}$ for all $\nu \vdash n$.

The above discussion shows that proving equality in the representation ring can be viewed as a linear algebra problem. In the case in which these vectors $\mathbf{a} = (a_\lambda)_{\lambda \vdash n}$ and $\mathbf{b} = (b_\lambda)_{\lambda \vdash n}$ have coordinates equal to 0 below a certain point, it will be convenient to further simplify the problem. We now make this more precise. Fix a positive integer k . Let $\pi_k(\mathbf{a}), \pi_k(\mathbf{b}), \pi_k(K), \pi_k(N)$ denote the submatrices obtained from $\mathbf{a}, \mathbf{b}, K, N$ by taking the only those entries corresponding to partitions with $\leq k$ parts. For K and N , this refers to entries whose row and column indices are partitions with $\leq k$ parts. (Intuitively, this corresponds to taking the ‘‘top parts’’ of \mathbf{a}, \mathbf{b} and the ‘‘upper-left corners’’ of K and N .)

LEMMA 2.15. *Let $\sum a_\lambda M^\lambda, \sum b_\lambda M^\lambda$ be elements in $\mathcal{R}ep(\mathfrak{S}_n)$. Let k be a positive integer. Suppose that $a_\lambda = b_\lambda = 0$ for all $\lambda \vdash n$ with more than k parts. Then the following are equivalent:*

- (1) $\sum a_\lambda M^\lambda = \sum b_\lambda M^\lambda$,
- (2) $\pi_k(N)\pi_k(\mathbf{a}) = \pi_k(N)\pi_k(\mathbf{b})$, and
- (3) $\sum a_\lambda \dim(M^\lambda)^{\mathfrak{S}_\nu} = \sum b_\lambda \dim(M^\lambda)^{\mathfrak{S}_\nu}$ for all $\nu \vdash n$ with $\leq k$ parts.

Sketch of the proof. It is straightforward to see that the essential claim is that $\pi_k(N)$ (the “upper-left corner” of N) is invertible. Recall that $N = K^T K$ and K is upper-triangular. Hence $\pi_k(N) = \pi_k(K^T)\pi_k(K)$. Moreover, $\pi_k(K^T) = \pi_k(K)^T$ and since K is upper-triangular with 1’s along the diagonal, the upper-left corner $\pi_k(K)$ has the same properties, and in particular is invertible. Thus $\pi_k(N)$ is also invertible. \square

Finally, we recall a fundamental result of Brosnan and Chow which identifies the dimension of the subspaces $H^*(\mathcal{Hess}(S, h))^{\mathfrak{S}_\nu}$ with the dimension of the cohomology of a regular Hessenberg variety and which we use repeatedly below.

THEOREM 2.16 ([3, Theorem 127]). *Let n be a positive integer and $h : [n] \rightarrow [n]$ a Hessenberg function. Let $\nu \vdash n$ be a partition of n , X_ν a regular operator of Jordan type ν , and S a regular semisimple operator. Then for each non-negative integer i ,*

$$\dim(H^{2i}(\mathcal{Hess}(S, h)))^{\mathfrak{S}_\nu} = \dim H^{2i}(\mathcal{Hess}(X_\nu, h)).$$

2.3. INCOMPARABILITY GRAPHS, ACYCLIC ORIENTATIONS, AND STANLEY’S THEOREM. In this section we recall some graph-theoretic data which can be constructed from a Hessenberg function. Let n be a positive integer and suppose $h : [n] \rightarrow [n]$ is a Hessenberg function.

DEFINITION 2.17. *We define the incomparability graph $\Gamma_h = (V(\Gamma_h), E(\Gamma_h))$ associated to h as follows. The vertex set $V(\Gamma_h)$ is $[n] = \{1, 2, \dots, n\}$. The edge set $E(\Gamma_h)$ is defined as follows: $\{i, j\} \in E(\Gamma_h)$ if $1 \leq j < i \leq n$ and $i \leq h(j)$.*

The incomparability graph is a visual representation of the set of roots Φ_h^- . In particular, it is straightforward to see that the roots of Φ_h^- are in bijection with the edges of Γ_h .

EXAMPLE 2.18. The incomparability graphs for $h = (2, 4, 4, 4)$ and $h = (3, 4, 5, 5, 5)$ are given below.



Recall that an *orientation* ω of (the edges of) a graph is an assignment of a direction (i.e. orientation) to each edge $e \in E(\Gamma_h)$. Equivalently, ω assigns to each edge e a source and a target; we denote the source (respectively target) of e according to the orientation ω by $\text{src}_\omega(e)$ (respectively $\text{tgt}_\omega(e)$). A (directed) *cycle* is a sequence of vertices starting and ending at the same vertex whose edges are oriented consistently with the order of the vertices in the sequence. We say that an orientation ω is *acyclic* if there are no (directed) cycles in the corresponding oriented graph. Let

$$\mathcal{A}(\Gamma_h) := \{\omega \mid \omega \text{ is an acyclic orientation of } \Gamma_h\}$$

denote the set of all acyclic orientations of Γ_h . Moreover, given an orientation ω , a *sink* associated to ω is a vertex v of the graph such that, for all edges e incident to the vertex v , the edge “points towards v ”, i.e. $\text{tgt}_\omega(e) = v$ for all edges e incident to v . It will turn out to be extremely important to pay close attention to the *number* of sinks associated to a given orientation. Thus we define

$$(11) \quad \mathcal{A}_k(\Gamma_h) := \{\omega \in \mathcal{A}(\Gamma_h) \mid \omega \text{ has exactly } k \text{ sinks}\}.$$

Since every acyclic orientation has at least one sink [21, Section 8.6, Exercise 4], we have $\mathcal{A}(\Gamma_h) = \bigsqcup_{k \geq 1} \mathcal{A}_k(\Gamma_h)$.

The following is a result of Shareshian and Wachs [22, Theorem 5.3] which generalizes a theorem of Stanley. Following their terminology, for an orientation ω of Γ_h , we let

$$(12) \quad \text{asc}(\omega) := |\{e = \{a, b\} \in E(\Gamma_h) \mid \text{src}_\omega(e) = a, \text{tgt}_\omega(e) = b, \text{ and } a < b\}|.$$

In other words, if Γ_h is drawn as in Example 2.18 with the labels of the vertices increasing from left to right, then $\text{asc}(\omega)$ is the total number of edges which “point to the right”.

THEOREM 2.19 ([22, Theorem 5.3]). *Let n be a positive integer and $h : [n] \rightarrow [n]$ a Hessenberg function. Let $c_{\lambda,i}$ denote the coefficients appearing in (9). Then for each i ,*

$$\sum_{\lambda \vdash n, \text{parts}(\lambda)=k} c_{\lambda,i} = |\{\omega \mid \omega \in \mathcal{A}_k(\Gamma_h) \text{ and } \text{asc}(\omega) = i\}|.$$

Since there is only one partition of n with exactly 1 part, namely $\lambda = (n)$, and because the representation $M^{(n)}$ corresponding to this partition is the trivial representation [8], we may immediately conclude the following, which will be important to us later on.

COROLLARY 2.20. *Under the conditions in the above theorem, the multiplicity of the trivial representation in $H^{2i}(\text{Hess}(\mathbf{S}, h))$ is the number of acyclic orientations ω of Γ_h with exactly 1 sink such that $\text{asc}(\omega) = i$. Equivalently, for $\lambda = (n)$ the trivial partition, we have*

$$c_{(n),i} = |\{\omega \mid \omega \in \mathcal{A}_1(\Gamma_h) \text{ and } \text{asc}(\omega) = i\}|.$$

The following is also immediate from Theorem 2.19 by summing over k .

COROLLARY 2.21. *Under the conditions in the above theorem we have*

$$\sum_{\lambda \vdash n} c_{\lambda,i} = |\{\omega \mid \omega \in \mathcal{A}(\Gamma_h) \text{ and } \text{asc}(\omega) = i\}|.$$

2.4. P_h -TABLEAUX AND A RESULT OF GASHAROV. Recall that the goal of the present manuscript is to prove a result about the coefficients c_λ and $c_{\lambda,i}$ appearing in (9), for certain cases of Hessenberg functions h . We have also seen that the coefficients d_λ from (6) are intimately related to the c_λ . In preparation for the arguments in the following sections, we now take a moment to recall a combinatorial object called a P_h -tableau, and the result of Gasharov which computes the d_λ 's in terms of P_h -tableaux.

DEFINITION 2.22. *Fix a Hessenberg function $h : [n] \rightarrow [n]$. A P_h -tableau of shape λ is a filling of a Young diagram of shape $\lambda \vdash n$ with the integers of $[n]$ such that*

- (1) *each integer $1, 2, \dots, n$ appears exactly once,*
- (2) *if $i \in [n]$ appears immediately to the right of $j \in [n]$ then $i > h(j)$, and*
- (3) *if $i \in [n]$ appears immediately below $j \in [n]$ then $j \leq h(i)$.*

EXAMPLE 2.23. Let $n = 5$ and let $h = (2, 3, 4, 5, 5)$. Then there are nine P_h -tableaux of shape $(2, 2, 1)$:

1	3	1	4	1	3	1	4	1	5	2	4	2	4	2	5	3	5
2	4	2	5	2	5	3	5	2	4	1	3	1	5	1	4	2	4
5		3		4		2		3		5		3		3		1	

Recall that every partition $\lambda \vdash n$ has a dual partition λ^\vee whose Young diagram is the transpose of the Young diagram of λ . The following theorem, which gives a positive, combinatorial formula for the coefficients d_λ , is due to Gasharov [9]. There is also a graded version of the theorem, due to Shareshian and Wachs [22, Theorem 6.3], but we will only need the ungraded version below.

THEOREM 2.24. *Let n be a positive integer and let $h : [n] \rightarrow [n]$ be a Hessenberg function. Let d_λ denote the coefficients appearing in (6). Then*

$$d_\lambda = |\{ P_h\text{-tableaux of shape } \lambda^\vee \}|.$$

3. ABELIAN HESSENBERG VARIETIES

In the previous sections we outlined the motivation behind this paper and recalled some background. We are finally ready to begin our own arguments in earnest, and the first task is to establish the terminology (and hypothesis) which allows us to make our arguments – namely, the definition of an abelian ideal and an abelian Hessenberg variety. We also briefly discuss how our special case relates to other situations that have been studied previously in the literature.

In Section 2 we defined an ideal of Φ^- associated to a Hessenberg function h . We now introduce the definition which is central to this manuscript.

DEFINITION 3.1. *We say that an ideal $I \subseteq \Phi^-$ is abelian if $\alpha + \beta \notin \Phi^-$ for all $\alpha, \beta \in I$.*

The notion of abelian ideals is not new in the context of Lie theory. However, as far as we are aware, its use in the study of Hessenberg varieties is new. The following definition is not essential to this paper but we include it because it frequently arises in the literature.

DEFINITION 3.2. *Let I be an ideal in Φ^- . We say that I is strictly negative if $-\Delta \cap I$ is empty.*

Note that if $I = I_h$ is the ideal of Φ^- associated to a Hessenberg function h , then $-\Delta \cap I$ is empty if and only if $-\Delta \subseteq \Phi_h$. The following is well-known, which partly explains why it is common practice in the study of Hessenberg varieties to assume that I_h is strictly negative.

LEMMA 3.3 ([19, Theorem 3.4]). *Let h be a Hessenberg function and $X \in \mathfrak{gl}(n, \mathbb{C})$ be a semisimple matrix. Then the corresponding semisimple Hessenberg variety $\text{Hess}(X, h)$ is connected if and only if I_h is strictly negative.*

EXAMPLE 3.4. In the case $n = 4$, there are 8 abelian ideals in Φ^- . The reader may check that these correspond to the Hessenberg functions $(1, 4, 4, 4)$, $(2, 2, 4, 4)$, $(2, 3, 4, 4)$, $(2, 4, 4, 4)$, $(3, 3, 3, 4)$, $(3, 3, 4, 4)$, $(3, 4, 4, 4)$ and $(4, 4, 4, 4)$. Among these, those that are strictly negative are

$$(4, 4, 4, 4), (3, 4, 4, 4), (3, 3, 4, 4), (2, 4, 4, 4), (2, 3, 4, 4).$$

and their corresponding ideals I_h are, respectively,

$$\emptyset, \{t_4 - t_1\}, \{t_4 - t_1, t_4 - t_2\}, \{t_4 - t_1, t_3 - t_1\}, \{t_4 - t_1, t_4 - t_2, t_3 - t_1\}.$$

The following extends the notion of abelian ideals to their corresponding Hessenberg varieties.

DEFINITION 3.5. *We say that the Hessenberg variety $\text{Hess}(X, h)$ and the corresponding Hessenberg function h are abelian, if I_h is abelian.*

Recall that Hessenberg functions are in bijection with natural unit interval orders as shown by Shareshian and Wachs in [22, Proposition 4.1]. Under this identification, the Hessenberg function h is abelian if and only if the longest chain (i.e. totally ordered subset) of the associated natural unit interval order has length one. We thank Timothy Chow for the following remark.

REMARK 3.6. There is also a purely combinatorial characterization of abelian Hessenberg functions as follows. Define the *index* $\text{index}(h)$ of a Hessenberg function to be the largest integer i such that $h(i) < n$. Then h is abelian if and only if $h(1) \geq \text{index}(h)$. Indeed, if i is the index of h and $h(1) < i$, then $t_i - t_1, t_n - t_i \in I_h$ since $h(1) < i \leq h(i) < n$ and $(t_i - t_1) + (t_n - t_i) = t_n - t_1 \in \Phi^-$ implying I_h is not abelian. On the other hand, if I_h is not abelian, then there exist roots $t_j - t_k, t_k - t_\ell \in I_h$, so

that $j > k > \ell$ and $j > h(k)$ and $k > h(\ell)$. Now, since $h(k) < j \leq n$, the index i of h is at least k , so we conclude that $h(1) \leq h(\ell) < k \leq i$.

Abelian ideals of Φ^- (or equivalently, of Φ^+) are the source of many combinatorial and Lie-theoretic formulas. The number of abelian ideals in the negative roots Φ^- of $\mathfrak{gl}(n, \mathbb{C})$ grows exponentially in n . This is a special case of a result by D. Peterson, as recorded by Kostant in [16, Theorem 2.1].

PROPOSITION 3.7 (Peterson, [16, Theorem 2.1]). *Let \mathfrak{g} be any semisimple Lie algebra and let Φ^- denote its set of negative roots. Then there are exactly $2^{\text{rk}(\mathfrak{g})}$ abelian ideals in Φ^- , where $\text{rk}(\mathfrak{g})$ denotes the rank of \mathfrak{g} .*

REMARK 3.8. We can also ask: how many ideals in Φ^- are both abelian and strictly negative? Suppose $h : [n] \rightarrow [n]$ is a Hessenberg function such that its corresponding ideal $I = I_h$ is abelian. Suppose in addition that I_h is not strictly negative. Then the corresponding Hessenberg space H is a maximal, standard parabolic Lie subalgebra of $\mathfrak{sl}(n, \mathbb{C})$. In particular,

$$|\{ \text{abelian ideals of } \Phi^- \text{ which are strictly negative} \}| = 2^{n-1} - (n - 1).$$

The set of abelian, strictly-negative Hessenberg varieties contains examples studied previously. For instance, suppose n is positive and $n \geq 3$. For any strictly negative Hessenberg function of the form $h = (m_1, m_2, n, n, \dots, n)$, Shareshian and Wachs proved results on the associated chromatic quasisymmetric polynomial which, when paired with Brosnan and Chow’s Theorem 2.10, proves Conjecture 2.8 for that case. It is not difficult to see that such a Hessenberg function is abelian using the alternative definition of an abelian Hessenberg function given in Remark 3.6.

In addition, the representation $H^*(\mathcal{Hess}(\mathbb{S}, h))$ for the Hessenberg function h associated to any standard parabolic subalgebra \mathfrak{p} was determined to be a direct sum of M^λ ’s by Teff [28, Theorem 4.20]. When \mathfrak{p} is a maximal parabolic subalgebra, the corresponding Hessenberg function is abelian so this is another special case of our result.

EXAMPLE 3.9. Consider $h = (3, 4, 5, 6, 6, 6)$, as in Example 2.3. This Hessenberg function corresponds to the abelian ideal

$$I_h = \{t_4 - t_1, t_5 - t_1, t_5 - t_2, t_6 - t_1, t_6 - t_2, t_6 - t_3\}.$$

However h is not of the form $(m_1, m_2, 6, 6, 6, 6)$, nor is it the Hessenberg function corresponding to a standard parabolic subalgebra. Therefore h is an example of a Hessenberg function for which our proof of Conjecture 2.8 is new.

4. SINK SETS, MAXIMUM SINK SET SIZE, AND AN INDUCTIVE DESCRIPTION OF ACYCLIC ORIENTATIONS

The results in Section 2.3 make it evident that the set of acyclic orientations, and the cardinalities of the sink sets associated to them, play a crucial role in determining the coefficients $c_{\lambda,i}$. The contribution of this manuscript is to further develop this circle of ideas by analyzing the sink sets themselves. We also pay close attention to those sinks sets which are of maximal cardinality. Below, we make these ideas more precise.

4.1. SINK SETS AND INDUCED SUBGRAPHS. Our first lemma gives several equivalent characterizations of a subset of $V(\Gamma_h)$ which can appear as a set of sinks for some acyclic orientation. We prepare some terminology. First, for a fixed Γ_h and acyclic orientation ω of Γ_h , let

$$\text{sk}(\omega) := \{v \in V(\Gamma_h) \mid v \text{ is a sink of } \omega\}$$

denote the set of sinks of ω . We say $\text{sk}(\omega)$ is a *sink set*. Recall that an *independent set* of vertices in Γ_h is a subset of $V(\Gamma_h)$ such that no two of them are connected by an edge in Γ_h . An independent set of vertices in the graph Γ_h corresponds to a chain in the associated natural unit interval order. We have the following.

LEMMA 4.1. *Let $h : [n] \rightarrow [n]$ be a Hessenberg function and Γ_h be the associated incomparability graph. Let $T = \{\ell_1 < \ell_2 < \dots < \ell_k\}$ be a subset of $V(\Gamma_h)$ for k a positive integer. Then the following are equivalent:*

- (1) T is a sink set, i.e. there exists an acyclic orientation $\omega \in \mathcal{A}_k(\Gamma_h)$ such that $T = \text{sk}(\omega)$,
- (2) $\ell_{i+1} > h(\ell_i)$ for all $i \in [k - 1]$, and
- (3) T is an independent set in Γ_h .

In particular, the cardinality of any maximum independent subset of vertices is equal to the cardinality of any maximum sink set.

Proof. We first show that (1) implies (2). Suppose T is a sink set. We wish to show $\ell_{i+1} > h(\ell_i)$ for all i , $1 \leq i \leq k - 1$. If $k = 1$, the condition is vacuous and there is nothing to check. If $k > 1$, suppose for a contradiction that there exists $i \in [k - 1]$ with $\ell_{i+1} \leq h(\ell_i)$. Then by construction of Γ_h there exists an edge e between ℓ_i and ℓ_{i+1} . For any orientation ω of Γ_h , we must have either $\text{tgt}_\omega(e) = \ell_i$ or $\text{tgt}_\omega(e) = \ell_{i+1}$, and not both. Thus ℓ_i and ℓ_{i+1} cannot be simultaneously contained in $\text{sk}(\omega)$, contradicting the fact that T is a sink set.

Next we prove that (2) implies (3). Note that since $\ell_1 < \ell_2 < \dots < \ell_k$ by assumption, if $\ell_{i+1} > h(\ell_i)$ for all $i \in [k - 1]$ then it follows that $\ell_b > h(\ell_a)$ for any pair $a < b$, $a, b \in [k]$. By construction of Γ_h this implies there are no edges in Γ_h connecting any two of the vertices in T . Hence, by definition, T is an independent set.

Finally we prove (3) implies (1). Suppose T is an independent set. Choose any total ordering of the vertices, beginning with the vertices in T . This total ordering uniquely determines an acyclic orientation ω of the Γ_h by requiring that $\text{tgt}_\omega(e)$ be the least vertex incident to e . Any such orientation satisfies the condition that $\text{sk}(\omega) = T$.

The last claim of the lemma follows immediately from the equivalence of (1) and (3). \square

As already mentioned, we wish to focus on the sink sets themselves, not just the acyclic orientations which give rise to them. Let $\mathcal{P}(V(\Gamma_h))$ be the power set of $V(\Gamma_h)$. We let

$$\text{SK}(\Gamma_h) := \{\text{sk}(\omega) \mid \omega \in \mathcal{A}(\Gamma_h)\} \subseteq \mathcal{P}(V(\Gamma_h))$$

denote the set of all subsets of $V(\Gamma_h)$ which can arise as the sink set of some acyclic orientation. Similarly, we let $\text{SK}_k(\Gamma_h)$ denote the subset of $\text{SK}(\Gamma_h)$ consisting of sink sets of cardinality k . By definition, we have

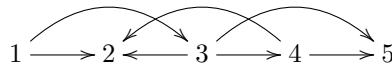
$$(13) \quad \mathcal{A}_k(\Gamma_h) = \bigsqcup_{T \in \text{SK}_k(\Gamma_h)} \{\omega \in \mathcal{A}_k(\Gamma_h) \mid \text{sk}(\omega) = T\}.$$

We call this the *sink set decomposition* and it is conceptually central to our later arguments. Indeed, recall that the statements of Theorems 2.19, Corollary 2.20, and Corollary 2.21 relate the coefficients $c_{\lambda,i}$ to cardinalities of certain subsets of $\mathcal{A}_k(\Gamma_h)$ for various k . Thus, the set of sink sets $\text{SK}_k(\Gamma_h)$ provides a way to further decompose $\mathcal{A}_k(\Gamma_h)$, and refine our understanding of the $c_{\lambda,i}$. More concretely, we now define a graph $\Gamma_h - T$ on $n - |T|$ vertices using the data of the original graph Γ_h together with the data of a sink set T . This construction will be critical for the induction argument in Sections 6 and 7.

Let $k \geq 1$ and suppose $T \in \text{SK}_k(\Gamma_h)$. Intuitively, the graph $\Gamma_h - T$ is obtained from Γ_h by “deleting” T . Moreover, we will identify this graph as the incomparability graph of a certain Hessenberg function h_T . We proceed in steps. The underlying set of vertices $V(\Gamma_h - T)$ is defined to be $V(\Gamma_h) \setminus T$. We also define the edge set $E(\Gamma_h - T)$ as follows: two vertices in $V(\Gamma_h - T)$ are connected in $\Gamma_h - T$ if and only if there exists an edge connecting them in $E(\Gamma_h)$. In graph theory $\Gamma_h - T$ is called the *induced subgraph* corresponding to the vertices $V(\Gamma_h) \setminus T \subseteq V(\Gamma_h)$.

In what follows it will sometimes be useful to label the vertices of $\Gamma_h - T$ by the integers $\{1, 2, \dots, n - k\} = [n - k]$ rather than $V(\Gamma_h) - T = [n] \setminus T$. Intuitively, this is straightforward: we simply re-label the vertices $V(\Gamma_h - T) = V(\Gamma_h) \setminus T$ by $[n - k]$ using the ordering induced by the ordering on the original $V(\Gamma_h)$. More precisely, for each $1 \leq j \leq n$, let j' denote the number of vertices $i \in T$ such that $i \leq j$. The function $\phi_T : [n] \rightarrow [n - k]$ defined by $\phi_T(j) = j - j'$ is surjective from $[n]$ to $[n - k]$ and restricts to a bijection between $[n] \setminus T$ and $[n - k]$.

EXAMPLE 4.2. Consider the graph Γ_h for $h = (3, 4, 5, 5, 5)$ and let $T = \{2, 5\}$. Then T is indeed a sink set, for the following acyclic orientation of Γ_h .



We also draw the (unoriented) graphs Γ_h and $\Gamma_h - T$ in the figure below. In the figure for Γ_h , the vertices of T and all incident edges to T are highlighted in red. The red edges and vertices are then deleted to obtain $\Gamma_h - T$ (with re-labelled vertices).



In this case, $\phi_T(1) = 1$, $\phi_T(3) = 2$, and $\phi_T(4) = 3$.

Using the above bijection $\phi_T : [n] \setminus T \rightarrow [n - k]$ we may define a function $h_T : [n - k] \rightarrow [n - k]$ by setting

$$(14) \quad h_T(\phi_T(i)) := \phi_T(h(i))$$

for all $i \in [n] \setminus T$. Our next claim is that the smaller graph $\Gamma_h - T$ is in fact the incomparability graph of h_T , so $\Gamma_h - T \cong \Gamma_{h_T}$. Note that in Example 4.2, the above construction yields the Hessenberg function $h_T = (2, 3, 3)$.

A reader familiar with the language of unit interval orders may note that the lemma below states that the result of deleting a chain from a unit interval order is still a unit interval order. Although the statement is clear in that context, we include a proof below in the language of Hessenberg functions for those readers who are not familiar with unit interval orders.

LEMMA 4.3. *Let $T \in \text{SK}_k(\Gamma_h)$ and let $\Gamma_h - T$ and h_T be defined as above. Then h_T is a Hessenberg function, and $\Gamma_h - T = \Gamma_{h_T}$.*

Proof. First we show that h_T is a Hessenberg function. Since ϕ_T and its inverse $\phi_T^{-1} : [n - k] \rightarrow [n] \setminus T$ are non-decreasing and h is non-decreasing by definition of Hessenberg functions, it follows that h_T is also non-decreasing. Next, the facts that $h(i) \geq i$ and ϕ_T is non-decreasing imply that

$$h_T(\phi_T(i)) = \phi_T(h(i)) \geq \phi_T(i)$$

for all $i \in [n] \setminus T$. Therefore h_T is a Hessenberg function.

Second, we wish to show that $\Gamma_h - T$ is the incomparability graph of h_T . To do this, fix $i, j \in [n] \setminus T$ satisfying $i < j$. It suffices to show that in $\Gamma_h - T$, there exists an edge between $\phi_T(i)$ and $\phi_T(j)$ if and only if $\phi_T(j) \leq h_T(\phi_T(i)) = \phi_T(h(i))$. Note there exists an edge between $\phi_T(i)$ and $\phi_T(j)$ in $\Gamma_h - T$ if and only if there

exists an edge between i and j in Γ_h by definition of $\Gamma_h - T$. This holds if and only if $j \leq h(i)$ by definition of Γ_h . This implies $\phi_T(j) \leq \phi_T(h(i))$ since ϕ_T is non-decreasing. On the other hand, given that $\phi_T(j) \leq \phi_T(h(i))$ we must show $j \leq h(i)$. Suppose for a contradiction that $j > h(i)$. Then we must have $\phi_T(h(i)) \leq \phi_T(j)$. This means $\phi_T(j) = \phi_T(h(i))$. Since ϕ_T is injective on $[n] \setminus T$ and $j \in [n] \setminus T$, this means $h(i) \in T$. Moreover, it follows from the definition of ϕ_T that $h(i) > j$. This contradicts the initial assumption that $j > h(i)$, so we conclude $j \leq h(i)$ as desired. Finally, $\phi_T(j) \leq \phi_T(h(i))$ if and only if $\phi_T(j) \leq h_T(\phi_T(i))$ by definition of h_T , completing the proof. \square

We have already observed that the edges of an incomparability graph Γ_h associated to a Hessenberg function are in one-to-one correspondence with the set of negative roots in Φ_h^- . Our construction of a “smaller” graph $\Gamma_{h_T} \cong \Gamma_h - T$, suggests that there should be a correspondence between negative roots in $\Phi_{h_T}^-$ and a certain subset of Φ_h^- which is determined by T . We now make this precise. By Lemma 4.3, we may describe the roots $\Phi_{h_T}^-$ and the ideal I_{h_T} corresponding to h_T using those of h as follows:

$$\Phi_{h_T}^- = \{t_{\phi_T(i)} - t_{\phi_T(j)} \mid t_i - t_j \in \Phi_h^- \text{ and } i, j \notin T\}$$

and

$$I_{h_T} = \{t_{\phi_T(i)} - t_{\phi_T(j)} \mid t_i - t_j \in I_h \text{ and } i, j \notin T\}.$$

In our computations below, it will also be convenient to consider the subset of negative roots in Φ_h^- and I_h which correspond to $\Phi_{h_T}^-$ and I_{h_T} , respectively, under the map ϕ_T . We set the notation

$$\Phi_h^-[T] := \{t_i - t_j \mid t_i - t_j \in \Phi_h^- \text{ and } i, j \notin T\}$$

and

$$I_h[T] := \{t_i - t_j \mid t_i - t_j \in I_h \text{ and } i, j \notin T\}.$$

There is an obvious bijection from $\Phi_h^-[T]$ to $\Phi_{h_T}^-$ and $I_h[T]$ to I_{h_T} given by $t_i - t_j \mapsto t_{\phi_T(i)} - t_{\phi_T(j)}$.

Finally, we observe that the construction of the smaller graph $\Gamma_h - T \cong \Gamma_{h_T}$ from the data of Γ_h also extends to orientations. Specifically, let $\omega \in \mathcal{A}_k(\Gamma_h)$ be any acyclic orientation such that $\text{sk}(\omega) = T$. Then the orientation ω naturally induces, by restriction, an orientation on $\Gamma_h - T = \Gamma_{h_T}$ (since the edges of $\Gamma_h - T$ are a subset of those of Γ_h). We denote this acyclic orientation on Γ_{h_T} by ω_T .

EXAMPLE 4.4. We continue with Example 4.2. In the pictures below, we draw an orientation ω of Γ_h on the left, and its corresponding induced orientation ω_T of Γ_{h_T} on the right. For visualization purposes the sink set T and its incident edges are highlighted in red.



4.2. SINK SETS OF MAXIMAL CARDINALITY AND AN INDUCTIVE DESCRIPTION OF ACYCLIC ORIENTATIONS. The main observation of the present section, recorded in Proposition 4.10, is that if k is maximal, then the sets appearing on the RHS of the sink set decomposition (13) are in bijective correspondence with the set of all acyclic orientations corresponding to the graphs Γ_{h_T} for $T \in \text{SK}_k(\Gamma_h)$. Moreover, this natural bijection gives a tight relationship between the number of ascending edges $\text{asc}(\omega)$ of the orientation ω of the original graph Γ_h with the number $\text{asc}(\omega_T)$ of the induced orientation on the smaller graph Γ_{h_T} , where ω_T is described at the end of Section 4.1 above. These ascending edge statistics record the degree – i.e. the grading

in $H^*(\text{Hess}(S, h))$ – in Theorem 2.19 and Corollary 2.20, so it is this relation which allows us to prove our “graded” results in Section 7.2.

We begin by making precise the notion of a sink set of maximum possible size.

DEFINITION 4.5. We define the maximum sink-set size $m(\Gamma_h)$ to be the maximum of the cardinalities of the sink sets $\text{sk}(\omega)$ associated to all possible acyclic orientations of Γ_h , i.e.

$$(15) \quad m(\Gamma_h) = \max\{|\text{sk}(\omega)| \mid \omega \in \mathcal{A}(\Gamma_h)\}.$$

Note that the maximum clearly exists since $|\text{sk}(\omega)|$ is bounded above by n . Furthermore, by Lemma 4.1, the maximal sink-set size of Γ_h is also the cardinality of a maximum independent set of vertices in Γ_h .

EXAMPLE 4.6. Continuing Example 4.2, the sink set $T = \{2, 5\}$ given in that example is in fact maximal, i.e. $m(\Gamma_h) = 2$. Indeed, in this case any set of three vertices must have at least one edge incident with two of them, and thus cannot be independent. Finally, we note that for this orientation we have $\text{asc}(\omega) = 5$, i.e. there are 5 edges pointing to the right.

Let $m = m(\Gamma_h)$ be the maximum sink-set size for a fixed incomparability graph Γ_h and Hessenberg function h as in Definition 4.5. We need some terminology. Suppose $T \in \text{SK}(\Gamma_h)$. Any acyclic orientation ω with sink set T must have some number of edges oriented to the right, as determined by the vertices in T .

DEFINITION 4.7. Suppose $T \in \text{SK}(\Gamma_h)$. We define the degree of T to be

$$\text{deg}(T) := \min\{\text{asc}(\omega) \mid \omega \in \mathcal{A}(\Gamma_h), \text{sk}(\omega) = T\}.$$

The next lemma shows that in practice it is easy to compute $\text{deg}(T)$ for any $T \in \text{SK}(\Gamma_h)$. Suppose $T = \{\ell_1 < \ell_2 < \dots < \ell_k\}$ is an independent set. We explicitly construct an acyclic orientation ω of Γ_h with sink set precisely T as follows. We first consider the set of edges e in Γ_h which are incident to a vertex in T . Note that any such e is incident to only one vertex, say ℓ_i , in T , because T is independent. We assign an orientation to any such e by requiring $\text{tgt}_\omega(e) = \ell_i$. Next consider all edges in Γ_h which are not incident to any vertex in T . To any such edge $e = \{v, v'\}$ where $v < v'$ we assign the orientation which makes the edge “point to the left”; more precisely, $\text{tgt}_\omega(e) = v$. The above clearly defines an acyclic orientation ω on Γ_h .

LEMMA 4.8. Let $T \in \text{SK}(\Gamma_h)$. Then

$$(16) \quad \text{deg}(T) = |\{e = \{\ell, \ell'\} \in E(\Gamma_h) \mid \ell' \in T, \ell < \ell'\}|.$$

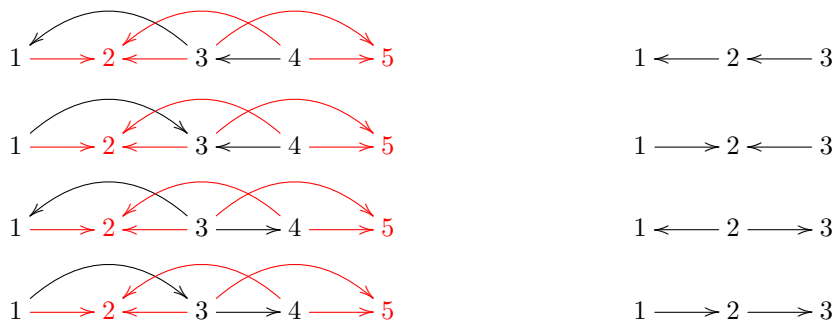
Moreover, $|\Phi_h^-| \geq |\Phi_{h_T}^-| + \text{deg}(T)$.

Proof. We begin with the first claim. Let ω denote the orientation of Γ_h constructed above. It can be verified that the edges which point to the right with respect to ω are precisely those which connect a vertex $\ell' \in T$ to a smaller vertex $\ell < \ell'$. Thus $\text{asc}(\omega)$ is equal to the RHS of (16) and hence $\text{deg}(T) \leq \text{RHS}$ by definition. To prove equality, we claim that for any $\omega' \in \mathcal{A}(\Gamma_h)$ with $\text{sk}(\omega') = T$ we must have $\text{asc}(\omega') \geq \text{asc}(\omega)$. But the fact that $\text{sk}(\omega') = T$ implies that any edge of the form $e = \{\ell, \ell'\}$ for $\ell < \ell'$ and $\ell' \in T$ must satisfy $\text{tgt}_{\omega'}(e) = \ell'$, i.e. e must point to the right. It follows that $\text{asc}(\omega') \geq \text{asc}(\omega)$ as desired.

For the second claim, recall that the edges of Γ_h (respectively Γ_{h_T}) are in bijection with Φ_h^- (respectively $\Phi_{h_T}^-$). By definition of h_T we know $|\Phi_h^-|$ is equal to $|\Phi_h^-[T]| = |\Phi_{h_T}^-|$ plus the total number of edges incident to any vertex of T . By (16), $\text{deg}(T)$ is less than or equal to the total number of vertices incident to a vertex in T so the claim follows. \square

Before stating our main proposition we illustrate with our running example.

EXAMPLE 4.9. Consider the graph Γ_h for $h = (3, 4, 5, 5, 5)$ as in Example 4.2. As already noted in Example 4.6, in this example we have $m(\Gamma_h) = 2$ and $T = \{2, 5\}$ is a sink set of maximal cardinality. In the figure below we draw all acyclic orientations $\omega \in \mathcal{A}(\Gamma_h)$ such that $\text{sk}(\omega) = \{2, 5\}$. The sink set T and incident edges are highlighted in red, and the corresponding acyclic orientation of Γ_{h_T} is displayed to the right.



In this example, we have $\text{deg}(T) = 3$.

In the example above, we obtain every acyclic orientation of Γ_{h_T} by restricting from an acyclic orientation of Γ_h with sink set T . The next proposition shows that this is always the case when T is not a strict subset of any other sink set. Note that this property may hold even if T does not have maximal cardinality. For example, the reader can check that $T = \{3\}$ in the ongoing example for $h = (3, 4, 5, 5, 5)$ satisfies this property despite the fact that $|T| = 1 < 2 = m(\Gamma_h)$.

PROPOSITION 4.10. Let $h : [n] \rightarrow [n]$ be a Hessenberg function and let $T \in \text{SK}_k(\Gamma_h)$ be a sink set which is not a strict subset of any other sink set. Then the restriction map

$$(17) \quad \{\omega \in \mathcal{A}_k(\Gamma_h) \mid \text{sk}(\omega) = T\} \rightarrow \mathcal{A}(\Gamma_{h_T}), \quad \omega \mapsto \omega_T$$

is a bijection. Furthermore, for any $\omega \in \mathcal{A}_k(\Gamma_h)$ with $\text{sk}(\omega) = T$ we have $\text{asc}(\omega) = \text{deg}(T) + \text{asc}(\omega_T)$.

Proof. We first claim that the given restriction map is injective. To see this, recall from Lemma 4.3 that $\Gamma_{h_T} = \Gamma_h - T$ and $\Gamma_h - T$ is obtained from Γ_h by deleting the vertices T and the edges incident to T . Now note that any orientation ω satisfying $\text{sk}(\omega) = T$ must have the property that $\text{tgt}_\omega(e) = v$ for any $v \in T$ and any edge e incident to v , by the definition of a sink. In other words, the orientation ω is determined on edges incident to T by the condition $\text{sk}(\omega) = T$. Thus the restriction map $\omega \mapsto \omega_T$, which forgets the orientations incident to T , is injective. Next we claim that the map is surjective. Recall that the equality $\Gamma_{h_T} = \Gamma_h - T$ identifies $V(\Gamma_{h_T})$ with $V(\Gamma_h) \setminus T$ and the edges of Γ_{h_T} with the edges in Γ_h which are not incident to T . Now suppose $\omega' \in \mathcal{A}(\Gamma_{h_T})$. Define an orientation ω'' on Γ_h by orienting all the edges incident to a vertex in T towards that vertex, and orienting the remaining edges using the orientation of ω' . We claim that ω'' is an acyclic orientation such that $\text{sk}(\omega'') = T$. This would show that the restriction map is surjective since $(\omega'')_T = \omega'$ by the definition of ω'' .

From the construction of ω'' it follows that $T \subseteq \text{sk}(\omega'')$. Since T is not a strict subset of any other sink set for Γ_h , it also follows that $\text{sk}(\omega'') = T$. To finish the proof it suffices to show that ω'' is acyclic. Suppose for a contradiction that ω'' contains an oriented cycle. Since $\omega' \in \mathcal{A}(\Gamma_{h_T})$, such a cycle must include at least one vertex v of T . An oriented cycle can contain a vertex only if that vertex has at least one edge oriented towards that vertex and one edge oriented outwards from that vertex. But

since v is a sink, this is impossible. Therefore no such oriented cycle can exist and $\omega'' \in \mathcal{A}(\Gamma_h)$ as desired.

Finally, consider $\omega \in \mathcal{A}_k(\Gamma_h)$ such that $\text{sk}(\omega) = T$ and recall that $\text{asc}(\omega)$ counts the number of edges which point to the right in ω . By Lemma 4.8, $\text{deg}(T)$ counts those edges incident to T that are oriented to the right and $\text{asc}(\omega_T)$ counts those edges oriented to the right in the induced subgraph on the vertices $V(\Gamma_h) - T$. Since these two sets of edges are mutually disjoint and together comprise all the edges of Γ_h pointing to the right, $\text{asc}(\omega) = \text{deg}(T) + \text{asc}(\omega_T)$ for all $\omega \in \{\omega \in \mathcal{A}_k(\Gamma_h) \mid \text{sk}(\omega) = T\}$ as desired. \square

5. THE HEIGHT OF AN IDEAL

In this section we introduce an integer invariant associated to an ideal I in Φ^- called the height of I . A (nonempty) ideal I is abelian precisely when its height is 1. In this sense, the case of abelian ideals is the “base case” of an inductive argument based on height. For a Hessenberg function h , we then observe a connection between the height of I_h and the maximum sink set size $m(\Gamma_h)$ defined in the previous section. This connection, together with the past results of Stanley, Shareshian–Wachs, and Gasharov as well as some standard representation theory, allows us to significantly simplify the process of proving Conjecture 2.8 in the abelian case.

Given an ideal $I \subseteq \Phi^-$, recall that we may form an ideal in the Borel subalgebra of lower triangular matrices in $\mathfrak{gl}(n, \mathbb{C})$. Namely, let $\mathcal{I} = \bigoplus_{\gamma \in I} \mathfrak{g}_\gamma$ be the ideal of root spaces corresponding to the roots in I . The *lower central series* of \mathcal{I} is the sequence of ideals defined inductively by

$$\mathcal{I}_1 = \mathcal{I}, \text{ and } \mathcal{I}_j = [\mathcal{I}, \mathcal{I}_{j-1}] \text{ for all } j \geq 2.$$

We define the lower central series of ideals in Φ^- analogously, by letting $I_j \subseteq \Phi^-$ be the unique ideal of negative roots such that $\mathcal{I}_j = \bigoplus_{\gamma \in I_j} \mathfrak{g}_\gamma$.

DEFINITION 5.1. *The height of an ideal $I \subseteq \Phi^-$, denoted $ht(I)$, is the length of its lower central series. More concretely,*

$$ht(I) := \max\{k \geq 1 \mid I_k \neq \emptyset\}.$$

If $I = \emptyset$, then we adopt the convention that $ht(I) = 0$. It is not hard to see that the height is well-defined, i.e. the maximum always exists.

As in the case of abelian ideals and Hessenberg functions, we can interpret the height of the ideal I_h using the language of unit interval orders. Indeed, the height of I_h is exactly the length of the longest chain in the natural unit interval order associated to h ; as proved in Proposition 5.8 below. Intuitively, the height of an ideal measures how “non-abelian” it is.

EXAMPLE 5.2. Consider the Hessenberg function $h = (2, 4, 4, 5, 5)$, with ideal

$$I_h = \{t_3 - t_1, t_4 - t_1, t_5 - t_1, t_5 - t_2, t_5 - t_3\}.$$

Then I_h is not abelian since $(t_3 - t_1) + (t_5 - t_3) = t_5 - t_1 \in \Phi^-$. We see that,

$$I_1 = I_h, I_2 = \{t_5 - t_1\}, \text{ and } I_j = \emptyset \text{ for all } j \geq 3$$

so $ht(I_h) = 2$.

To connect the notions of the height of the ideal I_h and maximal sink sets of Γ_h (or, equivalently, maximal chains in the associated natural unit interval order), we will use certain subsets of roots in Φ^- , defined below.

DEFINITION 5.3. Let $R \subseteq \Phi^-$. We say R is a subset of height k if there exist integers $q_1, q_2, \dots, q_k, q_{k+1} \in [n]$ such that $q_1 < q_2 < \dots < q_k < q_{k+1}$ and $R = \{t_{q_2} - t_{q_1}, t_{q_3} - t_{q_2}, \dots, t_{q_{k+1}} - t_{q_k}\}$. We let $\mathcal{R}_k(I)$ denote the set of all subsets of height k in an ideal I , and define $\mathcal{R}(I) := \bigsqcup_{k \geq 0} \mathcal{R}_k(I)$.

REMARK 5.4. It is straightforward to show that $R \subseteq \Phi^-$ is a subset of height k if and only if there exists $w \in \mathfrak{S}_n$ such that $w(R)$ is a subset of simple roots corresponding to k consecutive vertices in the Dynkin diagram for $\mathfrak{gl}(n, \mathbb{C})$.

If $R \subseteq \Phi^-$ is a subset of height k , then we may write $R = \{\beta_1, \beta_2, \dots, \beta_k\}$ where $\beta_i = t_{q_{i+1}} - t_{q_i}$ for the integers $q_1, q_2, \dots, q_k, q_{k+1} \in [n]$ given in the definition above. Therefore

$$(18) \quad \beta_i + \beta_{i+1} + \dots + \beta_j \in \Phi^- \text{ for all } 1 \leq i \leq j \leq k.$$

The next lemma proves a direct relationship between $ht(I)$ and subsets of maximal height in I .

LEMMA 5.5. Let I be a nonempty ideal in Φ^- . Then $ht(I) = \max\{|R| \mid R \in \mathcal{R}(I)\}$.

Proof. Let R be a subset of maximal height in I , so $|R| = k$ where $k = \max\{|R| \mid R \in \mathcal{R}(I)\}$. By definition, there exist $q_1, q_2, \dots, q_k, q_{k+1} \in [n]$ such that $q_1 < q_2 < \dots < q_k < q_{k+1}$ and $R = \{\beta_1, \beta_2, \dots, \beta_k\}$ where $\beta_i = t_{q_{i+1}} - t_{q_i}$ for each $1 \leq i \leq k$. Using the definition of the Lie bracket and [14, Proposition 8.5] we get that

$$\mathfrak{g}_{\beta_1 + \dots + \beta_i} = [\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_1 + \dots + \beta_{i-1}}] \subseteq [\mathcal{I}, \mathcal{I}_{i-1}] = \mathcal{I}_i$$

for all $2 \leq i \leq k$. In particular, $\mathfrak{g}_{\beta_1 + \dots + \beta_k} \subseteq \mathcal{I}_k$ so $I_k \neq \emptyset$ and therefore $ht(I) \geq k$.

Seeking a contradiction, suppose $ht(I) = k' > k$. Note that $k \geq 1$ since I is nonempty, so we have $k' \geq 2$. We claim that if this is the case, then $\mathcal{R}_{k'}(I) \neq \emptyset$, contradicting the assumption that k is maximal. Recall that I_i denotes the i -th ideal in the lower central series of $I = I_1$. By definition, if $ht(I) = k'$ then $I_{k'} \neq \emptyset$. Let $\gamma_{k'} \in I_{k'}$ so $\mathfrak{g}_{\gamma_{k'}} \subseteq \mathcal{I}_{k'} = [\mathcal{I}, \mathcal{I}_{k'-1}]$. By definition of the Lie bracket, there exists $\gamma_{k'-1} \in I_{k'-1}$ and $\alpha_{k'} \in I$ such that $\gamma_{k'} = \alpha_{k'} + \gamma_{k'-1}$. Applying the same reasoning, $\mathfrak{g}_{\gamma_{k'-1}} \subseteq \mathcal{I}_{k'-1} = [\mathcal{I}, \mathcal{I}_{k'-2}]$ so there exists $\gamma_{k'-2} \in I_{k'-2}$ and $\alpha_{k'-1} \in I$ such that $\gamma_{k'-1} = \alpha_{k'-1} + \gamma_{k'-2}$. Continue in this way to obtain $\gamma_i \in I_i$ for each $1 \leq i \leq k'$ and $\alpha_i \in I$ for each $2 \leq i \leq k'$ such that

$$(19) \quad \gamma_i = \alpha_i + \gamma_{i-1} \text{ for all } 2 \leq i \leq k'.$$

Set $\alpha_1 = \gamma_1$ and consider the set $R' = \{\alpha_1, \alpha_2, \dots, \alpha_{k'-1}, \alpha_{k'}\}$. For each i such that $1 \leq i \leq k'$, $\alpha_i \in \Phi^-$ so we may write $\alpha_i = t_{a_i} - t_{b_i}$ for some $a_i, b_i \in [n]$ such that $b_i < a_i$. We will prove the following claim.

CLAIM. Suppose i is an integer such that $2 \leq i \leq k'$. Then there exists an ordering $\{a'_1, a'_2, \dots, a'_i\}$ of the set $\{a_1, a_2, \dots, a_i\}$ so that if $\{b'_1, b'_2, \dots, b'_i\}$ is the corresponding re-ordering of the set $\{b_1, b_2, \dots, b_i\}$, i.e. the ordering so that

$$\{\alpha_1, \alpha_2, \dots, \alpha_i\} = \{t_{a'_1} - t_{b'_1}, t_{a'_2} - t_{b'_2}, \dots, t_{a'_i} - t_{b'_i}\}$$

holds, then $b'_j = a'_{j-1}$ for all $2 \leq j \leq i$.

Given this claim, consider the case in which $i = k'$ and let $q_1 = b'_1, q_2 = a'_1, \dots, q_{k'} = a'_{k'-1}$, and $q_{k'+1} = a'_{k'}$. We immediately get that $q_1 < q_2 < \dots < q_{k'} < q_{k'+1}$ since $q_j = a'_{j-1} = b'_j < a'_j = q_{j+1}$ for all $2 \leq j \leq k'$ and

$$R' = \{t_{q_2} - t_{q_1}, t_{q_3} - t_{q_2}, \dots, t_{q_{k'+1}} - t_{q_{k'}}\}.$$

Therefore $R' \in \mathcal{R}_{k'}(I)$, so $\mathcal{R}_{k'}(I) \neq \emptyset$, which is what we wanted to show.

We now prove the claim above, using induction on i . If $i = 2$, consider $\alpha_1 = t_{a_1} - t_{b_1}$ and $\alpha_2 = t_{a_2} - t_{b_2}$. Equation (19) implies that

$$(t_{a_1} - t_{b_1}) + (t_{a_2} - t_{b_2}) = \alpha_1 + \alpha_2 = \gamma_1 + \alpha_2 = \gamma_2 \in \Phi^-.$$

By definition of Φ^- we must have that either $a_2 = b_1$ or $a_1 = b_2$. If $a_2 = b_1$, set $a'_1 = a_2$ and $a'_2 = a_1$. The corresponding re-ordering of $\{b_1, b_2\}$ is $b'_1 = b_2$ and $b'_2 = b_1$. Then $b'_2 = a'_1$ as desired. Similarly, if $a_1 = b_2$, set $a'_1 = a_1$ and $a'_2 = a_2$. The corresponding re-ordering of $\{b_1, b_2\}$ is $b'_1 = b_1$ and $b'_2 = b_2$. In this case we also conclude that $b'_2 = a'_1$. Therefore the claim holds for $i = 2$.

Now assume that for some i such that $2 \leq i < k'$ there exists a reordering $\{a''_1, a''_2, \dots, a''_i\}$ of the set $\{a_1, a_2, \dots, a_i\}$ so that $b''_j = a''_{j-1}$ for all $2 \leq j \leq i$, where $\{b''_1, b''_2, \dots, b''_i\}$ is the corresponding re-ordering of the set $\{b_1, b_2, \dots, b_i\}$. By our assumptions,

$$\begin{aligned} (\alpha_1 + \alpha_2 + \dots + \alpha_i) + \alpha_{i+1} &= (t_{a''_1} - t_{b''_1} + t_{a''_2} - t_{a''_1} + \dots + t_{a''_i} - t_{a''_{i-1}}) + (t_{a_{i+1}} - t_{b_{i+1}}) \\ &= (t_{a''_i} - t_{b''_1}) + (t_{a_{i+1}} - t_{b_{i+1}}). \end{aligned}$$

On the other hand, Equation (19) implies that $(\alpha_1 + \alpha_2 + \dots + \alpha_i) + \alpha_{i+1} = \gamma_i + \alpha_{i+1} = \gamma_{i+1} \in \Phi^-$. It follows that either $a_{i+1} = b''_i$ or $a''_i = b_{i+1}$. If $a_{i+1} = b''_i$, set $a'_1 = a_{i+1}$ and $a'_j = a''_{j-1}$ for all $2 \leq j \leq i + 1$. The corresponding re-ordering of the set $\{b''_1, b''_2, \dots, b''_i\}$ is $b'_1 = b_{i+1}$ and $b'_j = b''_{j-1}$ for all $2 \leq j \leq i + 1$. It follows that $b'_2 = b''_1 = a_{i+1} = a'_1$ and for all $3 \leq j \leq i + 1$ we get $b'_j = b''_{j-1} = a''_{j-2} = a'_{j-1}$. Next, if $a''_i = b_{i+1}$, set $a'_j = a''_j$ for all $1 \leq j \leq i$ and $a'_{i+1} = a_{i+1}$. Using exactly the same reasoning as before (which is even simpler in this case since there are fewer shifts) we get that $b'_j = a'_{j-1}$ for all $2 \leq j \leq i + 1$, proving the induction hypothesis. \square

As mentioned above, the height of an ideal gives us another characterization of abelian ideals.

PROPOSITION 5.6. *Let $I \subseteq \Phi^-$ be a nonempty ideal. Then $\mathcal{R}_1(I)$ consists of all singleton sets of the roots in I . Moreover, I is abelian if and only if $ht(I) = 1$.*

Proof. Suppose I is a nonempty ideal. Each root in I is clearly a subset of height 1 in I . This proves the first claim. Now suppose I is abelian. This implies that there are no subsets of I of height 2 or more since any such subset would satisfy the summation conditions of (18). On the other hand, $\mathcal{R}_1(I) \neq \emptyset$ by the above argument. Therefore $ht(I) = 1$ by Lemma 5.5. For the converse, suppose $ht(I) = 1$. By definition of the lower central series, $\mathcal{I}_2 = [\mathcal{I}, \mathcal{I}] = \{0\}$. This means there cannot exist $\beta_1, \beta_2 \in I$ with $\beta_1 + \beta_2 \in \Phi^-$. Therefore I is abelian. \square

REMARK 5.7. The height of the ideal I_h is related to the *bounce number* and *bounce path* of a Dyck path, which also arise in the theory of Macdonald polynomials. Recall that there is a standard way to associate a Dyck path π_h to a Hessenberg function h [17, Proposition 2.7]. Given a Dyck path π_h , we can define the *bounce path* of π_h as in [13, Definition 3.1]. The bounce number of π_h is then the number of times its bounce path touches the diagonal. Indeed, it is not hard to see (for instance, using the characterization of abelian Hessenberg functions given in Remark 3.6) that I_h is abelian, i.e. the height of I_h is 1, if and only if this bounce number is 1.

Our next goal is to make the connection between the height of I_h and the maximal sink set size of Γ_h introduced in Section 4.2. More specifically, we assign to each sink set T of cardinality $k \geq 2$ a unique subset of roots in I_h of height $k - 1$ as follows. Suppose $T \in SK_k(\Gamma_h)$ where $k \geq 2$. Write T as $\{\ell_1, \ell_2, \dots, \ell_k\} \subseteq [n]$ where $\ell_1 < \ell_2 < \dots < \ell_k$. We define a function $SK_k(\Gamma_h) \rightarrow \mathcal{R}_{k-1}(I_h)$ by

$$(20) \quad T \mapsto R_T := \{\beta_i = t_{\ell_{i+1}} - t_{\ell_i} \mid 1 \leq i \leq k - 1\}.$$

We need the following.

PROPOSITION 5.8. *The map defined in (20) is well-defined, i.e. for any $T \in \text{SK}_k(\Gamma_h)$ with $k \geq 2$, the subset R_T is an element of $\mathcal{R}_{k-1}(I_h)$. Moreover, the map (20) is a bijection. In particular, $m(\Gamma_h) = \text{ht}(I_h) + 1$.*

Proof. Let $k \geq 2$ and $T \in \text{SK}_k(\Gamma_h)$, with $T = \{\ell_1, \dots, \ell_k\}$ as above. To see that (20) is well-defined we must first show that $R_T = \{\beta_i := t_{\ell_{i+1}} - t_{\ell_i} \mid 1 \leq i \leq k-1\}$ is a subset of I_h . By Lemma 4.1, $\ell_{i+1} > h(\ell(i))$ for all $i \in [k-1]$. Therefore each $\beta_i = t_{\ell_{i+1}} - t_{\ell_i} \in I_h$ as desired. The fact that R_T is a subset of height $k-1$ follows directly from the definition. We now claim (20) is a bijection. From the definition it is straightforward to see that it is injective, so it suffices to prove that it is also surjective. Suppose $R \in \mathcal{R}_{k-1}(I_h)$ is a set of height $k-1$. By definition there exist $q_1, q_2, \dots, q_{k-1}, q_k \in [n]$ such that $q_1 < q_2 < \dots < q_{k-1} < q_k$ and $R = \{t_{q_2} - t_{q_1}, t_{q_3} - t_{q_2}, \dots, t_{q_k} - t_{q_{k-1}}\}$. Consider the set $T = \{q_1, q_2, \dots, q_k\}$. Since $R \subseteq I_h$ we know $q_{i+1} > h(q_i)$ for all $i \in [k-1]$. Lemma 4.1 now implies that T is a sink set. By definition, $R = R_T$ is the image of T under (20) so our function is surjective. Our last assertion follows directly from the fact that $|\text{SK}_k(\Gamma_h)| = |\mathcal{R}_{k-1}(I_h)|$ for all $k \geq 2$ together with Lemma 5.5. This completes the proof. \square

The above lemma establishes, in particular, a bijection between $\text{SK}_2(\Gamma_h)$, the set of sink sets of size 2, and $\mathcal{R}_1(I_h)$. By Proposition 5.6 we know $\mathcal{R}_1(I_h) \cong I_h$ is the set of all singleton subsets of I_h , so this implies that

$$|\text{SK}_2(\Gamma_h)| = |I_h|.$$

More concretely, the bijection (20) associates to a sink set $T = \{j, i\} \subseteq [n]$ with $i > j$ the subset $\{t_i - t_j\} \subseteq I_h$ of height 1.

EXAMPLE 5.9. Continuing Example 4.2 with the acyclic orientation drawn therein, the sink set is $\{2, 5\}$ and the associated (singleton) subset of I_h of height 1 is $\{t_5 - t_2\} \subseteq I_h$.

Our next proposition makes the connection between the maximum sink-set size and the coefficients d_λ determining the representation $H^*(\text{Hess}(\mathbb{S}, h))$.

PROPOSITION 5.10. *Let $h : [n] \rightarrow [n]$ be a Hessenberg function. Then*

$$m(\Gamma_h) = \max\{i \mid d_\lambda \neq 0 \text{ for some } \lambda \vdash n \text{ with } i \text{ parts}\}$$

where the d_λ are the non-negative coefficients appearing in (6).

We first need the following lemma.

LEMMA 5.11. *If $T = \{i_1, i_2, \dots, i_k\}$ is a subset of $[n]$ whose elements fill a single row in a P_h -tableau, then T is an independent set of vertices in Γ_h .*

Proof. Suppose the elements of T are listed in increasing order (in the order they appear in the row of the P_h -tableau). By condition (2) in Definition 2.22, we get $i_j > h(i_{j-1})$ for all j such that $2 \leq j \leq k$. Lemma 4.1 now implies that T is an independent set of vertices. \square

Proof of Proposition 5.10. Let

$$\text{ind}(\Gamma_h) := \max\{|T| \mid T \subseteq V(\Gamma_h) \text{ and } T \text{ is independent}\}.$$

By Lemma 4.1 it suffices to show that

$$(21) \quad \text{ind}(\Gamma_h) = \max\{i \mid d_\lambda \neq 0 \text{ for some } \lambda \vdash n \text{ with } i \text{ parts}\}.$$

Suppose $\lambda \vdash n$ is a partition of n with k parts such that $d_\lambda \neq 0$. By Theorem 2.24 there exists at least one P_h -tableau of shape λ^\vee . Since λ has k parts, λ^\vee has k boxes

in the first row. By Lemma 5.11 the entries in the first row of this P_h -tableau form an independent set of vertices in Γ_h . Therefore the LHS of (21) is greater than or equal to the RHS.

To prove the opposite inequality, let $T = \{\ell_1, \ell_2, \dots, \ell_m\}$, where $\ell_1 < \ell_2 < \dots < \ell_m$ be an independent subset of vertices in Γ_h of maximal size. By Lemma 4.1 we know $\ell_{i+1} > h(\ell_i)$ for all $i \in [m - 1]$. Consider the partition $\lambda^\vee = (m, 1, \dots, 1)$ of n of “hook shape” with first row containing m boxes and all other rows containing only one box. Also consider the filling of the Young diagram of shape λ^\vee given by filling the top row with ℓ_1, \dots, ℓ_m in increasing order, and filling the remaining boxes by $[n] \setminus \{\ell_1, \dots, \ell_m\}$ in increasing order from top to bottom. We claim that this is a P_h -tableau of shape λ^\vee . By construction, conditions (1) and (2) of Definition 2.22 are already met, so we have only to check condition (3). Note that, for a pair i and j with i appearing immediately below j , the condition (3) (namely, that $j \leq h(i)$) holds automatically if $j < i$ (since $h(i) \geq i$ by definition of Hessenberg functions). Since λ^\vee is of hook shape, the only places where condition (3) must be checked is along the leftmost column of λ^\vee , and since by construction the filling contains entries which increase from top to bottom starting at the second row, the argument above implies that the only remaining place where condition (3) must be checked is for the entry ℓ_1 in the top-left box of λ^\vee and the entry $\ell' := \min([n] \setminus \{\ell_1, \dots, \ell_m\})$ in the unique box in the second row, for which we must show that $\ell_1 \leq h(\ell')$. Suppose for a contradiction that $\ell_1 > h(\ell')$ (and hence $\ell' < \ell_1$). This implies there is no edge connecting ℓ' with ℓ_1 for any $i, 1 \leq i \leq m$. Thus $T' = \{\ell', \ell_1, \dots, \ell_m\}$ is a sink set of Γ_h by Lemma 4.1. Since $|T'| = m + 1$, this contradicts the maximality of $m = |T|$. Thus $\ell_1 \leq h(\ell')$ and hence the above filling is indeed a P_h -tableau. By construction of the λ^\vee , its dual partition λ has $m \geq k + 1$ parts proving that the RHS of Equation (21) is greater than or equal to the LHS. \square

The following is now straightforward. In the case that I_h is abelian, the corresponding restriction on the partitions that can appear in the RHS of (9) is quite striking.

COROLLARY 5.12. *Let $h : [n] \rightarrow [n]$ be a Hessenberg function and let c_λ and $c_{\lambda,i}$ be the coefficients appearing in (9). Then $c_\lambda = c_{\lambda,i} = 0$ for all $\lambda \vdash n$ with more than $m(\Gamma_h) = ht(I_h) + 1$ parts and for all $i \geq 0$. In particular, if I_h is abelian, then $c_\lambda = c_{\lambda,i} = 0$ for all $\lambda \vdash n$ with more than 2 parts and for all $i \geq 0$.*

Proof. It follows from Proposition 5.10 that, under the hypotheses, $d_\lambda = 0$ for all λ with more than $m(\Gamma_h)$ parts. Now apply Lemma 2.12 to $H^*(\mathcal{Hess}(\mathbf{S}, h))$. For the abelian case, if I_h is non-empty then this follows from Propositions 5.6 and 5.8. If I_h is empty, then $h = (n, n, \dots, n)$ and $\mathcal{Hess}(\mathbf{S}, h) = \mathcal{Flags}(\mathbb{C}^n)$. The corresponding graph Γ_h has the property that every vertex is connected to every other vertex, implying that $m(\Gamma_h) = 1$ and hence $c_\lambda = c_{\lambda,i} = 0$ for all λ with 2 or more parts and all $i \geq 0$. Hence the conclusion holds in this case as well. \square

We have already indicated that our strategy for proving Theorem 1.1 is by induction, using the association of Γ_h with $\Gamma_{h_T} = \Gamma_h - T$ for a sink set T as in Lemma 4.3. Let \mathbf{S}_T denote any regular semisimple element in $\mathfrak{gl}(n - |T|, \mathbb{C})$. It will be useful for us to know that vanishing conditions on the coefficients of the dot action representation on $H^*(\mathcal{Hess}(\mathbf{S}, h))$ imply vanishing conditions for the analogous coefficients of $H^*(\mathcal{Hess}(\mathbf{S}_T, h_T))$. To state the lemma precisely we introduce some terminology. For each partition μ of $n - |T|$ we define $c_{\mu,i}^T$ (respectively $d_{\mu,i}^T$) to be the coefficient of M^μ (respectively \mathcal{S}^μ) for the decomposition of the $\mathfrak{S}_{n-|T|}$ -representation $H^{2i}(\mathcal{Hess}(\mathbf{S}_T, h_T))$ in $\mathcal{Rep}(\mathfrak{S}_{n-|T|})$.

LEMMA 5.13. *Let $h : [n] \rightarrow [n]$ be a Hessenberg function, and let $T \in \text{SK}_k(\Gamma_h)$ be a sink set of Γ_h . Then*

- (1) $m(\Gamma_{h_T}) \leq m(\Gamma_h)$ and
- (2) $c_\mu^T = 0$ and $c_{\mu,i}^T = 0$ for all $\mu \vdash (n - |T|)$ with more than $m(\Gamma_h)$ parts and all $i \geq 0$.

In particular, if I_h is abelian and $T \in \text{SK}_2(\Gamma_h)$, then $c_{\lambda,i}^T = 0$ for all λ with more than 2 parts and all $i \geq 0$.

Proof. We begin with the first claim. Since Γ_{h_T} is by definition an induced subgraph of Γ_h , if there exists an independent set of vertices in Γ_{h_T} , then the corresponding subset is also independent in Γ_h . It follows from Lemma 4.1 that $m(\Gamma_{h_T}) = \max\{|T'| \mid T' \subseteq V(\Gamma_{h_T}) \text{ is independent in } \Gamma_{h_T}\}$ so we conclude $m(\Gamma_{h_T}) \leq m(\Gamma_h)$ as desired. The second statement follows from the first by Corollary 5.12. \square

6. AN INDUCTIVE FORMULA FOR THE COEFFICIENTS OF THE DOT ACTION

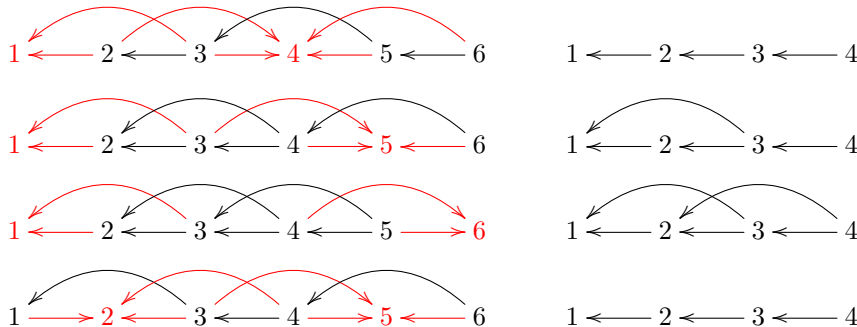
In this section, we state our main theorem, which gives an inductive formula which, in the case when I_h is abelian, expresses Tymoczko’s “dot action” representation on $H^{2i}(\text{Hess}(\mathbf{S}, h))$ as a combination of trivial representations together with a sum of tabloid representations with coefficients associated to smaller Hessenberg varieties in $\text{Flags}(\mathbb{C}^{n-2})$. To illustrate this result, we give an extended example when $n = 6$. We also state three technical results – one (simple) lemma and two propositions – and give a proof of Theorem 6.1 based on these three results. Each of the Propositions below are themselves inductive formulas, and are of interest in their own right. The proofs of the two propositions are postponed to Section 7.

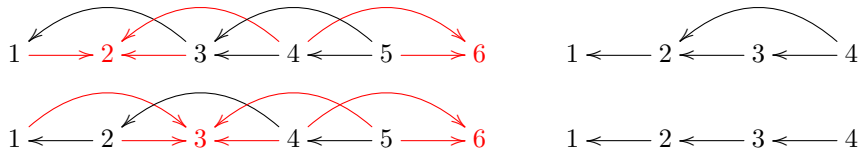
THEOREM 6.1. *Let n be a positive integer and $n \geq 3$. Let $h : [n] \rightarrow [n]$ be a Hessenberg function such that I_h is abelian and $i \geq 0$ be a non-negative integer. In the representation ring $\text{Rep}(\mathfrak{S}_n)$ we have the equality*

$$(22) \quad H^{2i}(\text{Hess}(\mathbf{S}, h)) = c_{(n),i} M^{(n)} + \sum_{T \in \text{SK}_2(\Gamma_h)} \left(\sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu,i-\text{deg}(T)}^T M^{(\mu_1+1, \mu_2+1)} \right).$$

We first illustrate the theorem via an extended example.

EXAMPLE 6.2. Let $n = 6$ and $h = (3, 4, 5, 6, 6, 6)$ as in Example 2.3. Then I_h is abelian, and $|I_h| = 6$. Thus, there are six maximum dimensional sink sets in $\text{SK}_2(\Gamma_h)$. The graphs below show the acyclic orientation $\omega \in \mathcal{A}_2(\Gamma_h)$ such that $\text{asc}(\omega) = \text{deg}(T)$ for each $T \in \text{SK}_2(\Gamma_h)$. In each case, the sink set T and incident edges are highlighted in red and we display the corresponding acyclic orientation of $\Gamma_h - T \cong \Gamma_{h_T}$ on the right.





Each of the graphs Γ_{h_T} in the right column above corresponds to one of the Hessenberg functions: $(2, 3, 4, 4)$, $(3, 3, 4, 4)$, $(3, 4, 4, 4)$, $(2, 4, 4, 4)$. Since the graphs are symmetric, $\Gamma \setminus \{1, 5\} \cong \Gamma \setminus \{2, 6\}$ and $\mathcal{Hess}(S', (3, 3, 4, 4)) \cong \mathcal{Hess}(S', (2, 4, 4, 4))$ where $S' \in \mathfrak{gl}(n - 2, \mathbb{C})$ is a regular semisimple element. The representation $H^*(\mathcal{Hess}(S', h_T))$ for each $T \in \text{SK}_2(\Gamma_h)$ is as shown in the table below. The reader can confirm this using the graded version of Theorem 2.24, namely [22, Theorem 6.3], together with (8).

Hessenberg function h_T :	$(2, 3, 4, 4)$	$(3, 3, 4, 4)$	$(3, 4, 4, 4)$
$H^0(\mathcal{Hess}(S', h_T))$	$M^{(4)}$	$M^{(4)}$	$M^{(4)}$
$H^2(\mathcal{Hess}(S', h_T))$	$M^{(4)} + M^{(3,1)} + M^{(2,2)}$	$2M^{(4)} + M^{(3,1)}$	$3M^{(4)}$
$H^4(\mathcal{Hess}(S', h_T))$	$M^{(4)} + M^{(3,1)} + M^{(2,2)}$	$2M^{(4)} + 2M^{(3,1)}$	$4M^{(4)} + M^{(3,1)}$
$H^6(\mathcal{Hess}(S', h_T))$	$M^{(4)}$	$2M^{(4)} + M^{(3,1)}$	$4M^{(4)} + M^{(3,1)}$
$H^8(\mathcal{Hess}(S', h_T))$		$M^{(4)}$	$3M^{(4)}$
$H^{10}(\mathcal{Hess}(S', h_T))$			$M^{(4)}$

Next we see that $\deg(\{1, 4\}) = \deg(\{1, 5\}) = \deg(\{1, 6\}) = 2$, $\deg(\{2, 5\}) = \deg(\{2, 6\}) = 3$, and $\deg(\{3, 6\}) = 4$ from the graphs above. We now have all the information we need to compute $H^*(\mathcal{Hess}(S, h))$ in all degrees as the shifted sum of $M^{(\mu_1+1, \mu_2+1)}$'s where $M^{(\mu_1, \mu_2)}$ appears in the representations above. The next two tables show how to shift these representations using $\deg(T)$ in order to obtain $H^*(\mathcal{Hess}(S, h))$.

$T \in \text{SK}_2(\Gamma_h)$:	$\{1, 4\}$	$\{1, 5\}$	$\{1, 6\}$
$H^2(\mathcal{Hess}(S, h))$			
$H^4(\mathcal{Hess}(S, h))$	$M^{(5,1)}$	$M^{(5,1)}$	$M^{(5,1)}$
$H^6(\mathcal{Hess}(S, h))$	$M^{(5,1)} + M^{(4,2)} + M^{(3,3)}$	$2M^{(5,1)} + M^{(4,2)}$	$3M^{(5,1)}$
$H^8(\mathcal{Hess}(S, h))$	$M^{(5,1)} + M^{(4,2)} + M^{(3,3)}$	$2M^{(5,1)} + 2M^{(4,2)}$	$4M^{(5,1)} + M^{(4,2)}$
$H^{10}(\mathcal{Hess}(S, h))$	$M^{(5,1)}$	$2M^{(5,1)} + M^{(4,2)}$	$4M^{(5,1)} + M^{(4,2)}$
$H^{12}(\mathcal{Hess}(S, h))$		$M^{(5,1)}$	$3M^{(5,1)}$
$H^{14}(\mathcal{Hess}(S, h))$			$M^{(5,1)}$
$H^{16}(\mathcal{Hess}(S, h))$			

$T \in \text{SK}_2(\Gamma_h)$:	$\{2, 5\}$	$\{3, 6\}$	$\{2, 6\}$
$H^4(\mathcal{Hess}(S, h))$			
$H^6(\mathcal{Hess}(S, h))$	$M^{(5,1)}$		$M^{(5,1)}$
$H^8(\mathcal{Hess}(S, h))$	$M^{(5,1)} + M^{(4,2)} + M^{(3,3)}$	$M^{(5,1)}$	$2M^{(5,1)} + M^{(4,2)}$
$H^{10}(\mathcal{Hess}(S, h))$	$M^{(5,1)} + M^{(4,2)} + M^{(3,3)}$	$M^{(5,1)} + M^{(4,2)} + M^{(3,3)}$	$2M^{(5,1)} + 2M^{(4,2)}$
$H^{12}(\mathcal{Hess}(S, h))$	$M^{(5,1)}$	$M^{(5,1)} + M^{(4,2)} + M^{(3,3)}$	$2M^{(5,1)} + M^{(4,2)}$
$H^{14}(\mathcal{Hess}(S, h))$		$M^{(5,1)}$	$M^{(5,1)}$
$H^{16}(\mathcal{Hess}(S, h))$			

For example, we get,

$$H^8(\mathcal{Hess}(S, h)) = c_{(6),4}M^{(6)} + 11M^{(5,1)} + 6M^{(4,2)} + 2M^{(3,3)}.$$

As mentioned above, we prove Theorem 6.1 using the following three results, recorded as a Lemma and two Propositions, each of which are themselves inductive formulas. Indeed, Lemma 6.3 expresses the number $N_{\lambda',\lambda}$ associated to two partitions of n in terms of the same value associated to two partitions of the smaller integer $n - 2$. Proposition 6.5 gives a formula for the Poincaré polynomial of $\mathcal{H}ess(\mathbf{N}, h) \subseteq \mathcal{F}lags(\mathbb{C}^n)$ in terms of Poincaré polynomials of regular nilpotent Hessenberg varieties in $\mathcal{F}lags(\mathbb{C}^{n-2})$, and Proposition 6.6 is of a similar flavor. Throughout the remainder of this section and the next, for a positive integer $n \geq 3$, we let \mathbf{N}' and \mathbf{S}' denote choices of regular nilpotent and regular semisimple elements, respectively, in $\mathfrak{gl}(n - 2, \mathbb{C})$.

LEMMA 6.3. *Let n be a positive integer and $n \geq 3$. Let $\mu = (\mu_1, \mu_2)$ and $\mu' = (\mu'_1, \mu'_2)$ be any partitions of $n - 2$ with at most 2 parts. Then*

$$(23) \quad \dim \left(M^{(\mu_1+1, \mu_2+1)} \right)^{\mathfrak{S}^{(\mu'_1+1, \mu'_2+1)}} = \dim \left(M^\mu \right)^{\mathfrak{S}^{\mu'}} + 1.$$

Proof. Recall from (10) and the related discussion that $N_{\mu, \mu'} = \dim(M^\mu)^{\mathfrak{S}^{\mu'}}$ and the matrix $N = (N_{\mu, \mu'})$ is symmetric. To prove the lemma it clearly suffices to prove the formula

$$(24) \quad N_{(a,b),(c,d)} = b + 1$$

for any $a, b, c, d \geq 0$ integers with $a + b = c + d = k$ for a fixed positive integer k and $a \geq c$, since this would imply that the LHS and RHS of (23) are equal, thus proving (23). To prove (24), we recall that in general $N_{\mu, \mu'}$ is the number of matrices $A = (a_{ij})$ with $a_{ij} \geq 0$ integers such that $\text{row}(A) = \mu$ and $\text{col}(A) = \mu'$ (see [26, Corollary 7.12.3]), where $\text{row}(A)$ is the vector obtained from a matrix by taking row-wise sums, and $\text{col}(A)$ is the vector obtained by taking column-wise sums,

$$\text{row}(A) := (r_1, r_2, \dots)$$

where $r_i = \sum_j a_{ij}$ and

$$\text{col}(A) := (c_1, c_2, \dots)$$

where $c_j = \sum_i a_{ij}$. In our case, since both (a, b) and (c, d) have only 2 parts, this is equal to the number of matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ such that } \alpha + \beta = a, \gamma + \delta = b, \alpha + \gamma = c, \text{ and } \beta + \delta = d.$$

It is both straightforward to see and well-known that this is the number of ways to fill a Young diagram of shape (a, b) with c many 1's and d many 2's, such that the rows are weakly increasing. Since there are only 2 rows in this Young diagram, the filling is completely determined by the number of boxes in the 2nd row which contain a 1. Since $a \geq c$, it follows that $d \geq b$, and this number of boxes is between 0 and b . Thus there are precisely $b + 1$ many such fillings, proving (24) as desired. \square

REMARK 6.4. It is also a well known fact that $\dim(M^\lambda)^\mu = |\mathfrak{S}_\mu \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda|$. When both λ and μ have two parts, there is a set of coset representatives for $\mathfrak{S}_\mu \backslash \mathfrak{S}_n / \mathfrak{S}_\lambda$ known as *bigrassmanian permutations*. These elements play an important role in the combinatorial properties of the symmetric group and in Schubert calculus.

For any $\mathbf{X} \in \mathfrak{gl}(n, \mathbb{C})$ and Hessenberg function $h : [n] \rightarrow [n]$, we denote by $P(\mathcal{H}ess(\mathbf{X}, h), t)$ the Poincaré polynomial (with variable t) associated to the Hessenberg variety $\mathcal{H}ess(\mathbf{X}, h)$. For the varieties considered in this paper, all Poincaré polynomials are concentrated in even degrees.

PROPOSITION 6.5. *Let n be a positive integer and $n \geq 3$. Let $h : [n] \rightarrow [n]$ be a Hessenberg function such that I_h is abelian. Let \mathbf{N} be a regular nilpotent element of $\mathfrak{gl}(n, \mathbb{C})$ and \mathbf{N}' be a regular nilpotent element of $\mathfrak{gl}(n - 2, \mathbb{C})$. Then*

$$P(\mathcal{H}ess(\mathbf{N}, h), t) = \sum_{i=0}^{|\Phi_h^-|} c_{(n),i} t^{2i} + \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(\mathbf{N}', h_T), t).$$

In particular, the $2i$ -th Betti number of $\mathcal{H}ess(\mathbf{N}, h)$ satisfies

$$\dim H^{2i}(\mathcal{H}ess(\mathbf{N}, h)) = c_{(n),i} + \sum_{T \in \text{SK}_2(\Gamma_h)} \dim H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathbf{N}', h_T)).$$

PROPOSITION 6.6. *Let n be a positive integer, $n \geq 3$. Let $h : [n] \rightarrow [n]$ be a Hessenberg function such that I_h is abelian. Let \mathbf{X}_ν be the regular element of $\mathfrak{gl}(n, \mathbb{C})$ associated to $\nu = (\mu_1 + 1, \mu_2 + 1) \vdash n$ and \mathbf{X}_μ be a regular element of $\mathfrak{gl}(n - 2, \mathbb{C})$ associated to $\mu = (\mu_1, \mu_2) \vdash (n - 2)$. Then*

$$(25) \quad P(\mathcal{H}ess(\mathbf{X}_\nu, h), t) = P(\mathcal{H}ess(\mathbf{N}, h), t) + \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(\mathbf{X}_\mu, h_T), t).$$

In particular, the $2i$ -th Betti number of $\mathcal{H}ess(\mathbf{X}_\nu, h)$ satisfies

$$\begin{aligned} \dim H^{2i}(\mathcal{H}ess(\mathbf{X}_\nu, h)) \\ = \dim H^{2i}(\mathcal{H}ess(\mathbf{N}, h)) + \sum_{T \in \text{SK}_2(\Gamma_h)} \dim H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathbf{X}_\mu, h_T)). \end{aligned}$$

Below, we give a proof of Theorem 6.1 using the three results above. The basic idea of the proof is as follows. A priori, the assertion of Theorem 6.1 is an equality in the representation ring $\text{Rep}(\mathfrak{S}_n)$. We first reduce this problem to a collection of equalities of integers by taking \mathfrak{S}_ν -invariants for varying $\nu \vdash n$ and using Proposition 2.13. Next, we repeatedly use Brosnan and Chow’s Theorem 2.16 to relate these \mathfrak{S}_ν -invariant subspaces to the Betti numbers of other regular Hessenberg varieties. In this manner, the problem is reduced to an induction on the Poincaré polynomials of regular Hessenberg varieties.

Proof of Theorem 6.1. Since h is an abelian Hessenberg function, we know from Corollary 5.12 that $c_{\lambda,i} = 0$ for all $\lambda \vdash n$ with 3 or more parts. In other words, by the abelian assumption, we know

$$H^{2i}(\mathcal{H}ess(\mathbf{S}, h)) = \sum_{\substack{\lambda \vdash n \\ \lambda \text{ has at most 2 parts}}} c_{\lambda,i} M^\lambda.$$

Therefore the LHS of (22), can be written as a linear combination of M^λ ’s for λ with at most 2 parts. By inspection, the same is true of the RHS of (22). An application of Lemma 2.15 implies that in order to prove (22) it suffices to prove the equality

$$(26) \quad \dim(H^{2i}(\mathcal{H}ess(\mathbf{S}, h)))^{\mathfrak{S}_\nu} = c_{(n),i} \dim(M^{(n)})^{\mathfrak{S}_\nu} + \sum_{T \in \text{SK}_2(\Gamma_h)} \left(\sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu,i-\deg(T)}^T \dim(M^{(\mu_1+1, \mu_2+1)})^{\mathfrak{S}_\nu} \right)$$

for all $\nu \vdash n$ with at most 2 parts.

Note that since $M^{(n)}$ is the trivial 1-dimensional \mathfrak{S}_n -representation, we have $(M^{(n)})^{\mathfrak{S}_\nu} = M^{(n)}$ for all $\nu \vdash n$, and in particular, $\dim(M^{(n)})^{\mathfrak{S}_\nu} = 1$ for all $\nu \vdash n$. We also know from Theorem 2.16 that

$$\dim(H^{2i}(\mathcal{H}ess(\mathcal{S}, h)))^{\mathfrak{S}_\nu} = \dim H^{2i}(\mathcal{H}ess(\mathbf{X}_\nu, h))$$

where \mathbf{X}_ν denotes a regular element of $\mathfrak{gl}(n, \mathbb{C})$ with Jordan block sizes given by $\nu \vdash n$. It follows that it suffices to prove

$$(27) \quad \dim H^{2i}(\mathcal{H}ess(\mathbf{X}_\nu, h)) = c_{(n),i} + \sum_{T \in \text{SK}_2(\Gamma_h)} \left(\sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \deg(T)}^T \dim(M^{(\mu_1+1, \mu_2+1)})^{\mathfrak{S}_\nu} \right)$$

for all $\nu \vdash n$ with at most two parts.

To prove (27) we take cases. First, we consider (27) for the unique case in which $\nu \vdash n$ has only one part, namely $\nu = (n)$. In this case $\mathfrak{S}_\nu = \mathfrak{S}_n$ and \mathbf{X}_ν is the $n \times n$ nilpotent matrix in Jordan form with a single Jordan block. Recall also that $\dim(M^{(\mu_1+1, \mu_2+1)})^{\mathfrak{S}_n} = 1$ since the multiplicity of the trivial representation in any $M^{(\mu_1+1, \mu_2+1)}$ is 1. Thus, we first observe that the RHS of (27) is

$$(28) \quad c_{(n),i} + \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \deg(T)}^T$$

and second, to simplify (27) further, we recall that the coefficients $c_{\mu, i - \deg(T)}^T$ appearing there are associated to $H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathcal{S}', h_T))$ by the equality

$$H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathcal{S}', h_T)) = \sum_{\mu \vdash (n-2)} c_{\mu, i - \deg(T)}^T M^\mu.$$

Now the assumption that I_h is abelian implies $c_{\mu, i - \deg(T)}^T = 0$ for all $\mu \vdash (n-2)$ with more than 2 parts by Lemma 5.13. Thus we have

$$H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathcal{S}', h_T)) = \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \deg(T)}^T M^\mu$$

and taking \mathfrak{S}_{n-2} -invariants we obtain

$$\begin{aligned} \dim H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathbf{N}', h_T)) &= \dim(H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathcal{S}', h_T)))^{\mathfrak{S}_{n-2}} \\ &= \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \deg(T)}^T \end{aligned}$$

where the first equality follows from Theorem 2.16. Therefore (28) can be rewritten as

$$c_{(n),i} + \sum_{T \in \text{SK}_2(\Gamma_h)} \dim H^{2i-2 \deg(T)}(\mathcal{H}ess(\mathbf{N}', h_T))$$

and now (27) follows for the case $\nu = (n)$ and $\mathbf{X}_\nu = \mathbf{N}$ by Proposition 6.5.

Next, we consider the case in which $\nu = (\mu'_1 + 1, \mu'_2 + 1) \vdash n$ for some $\mu' = (\mu'_1, \mu'_2) \vdash (n-2)$, i.e. the case in which ν has exactly two parts. Using an argument

similar to the above, the RHS of (27) for $\nu = (\mu'_1 + 1, \mu'_2 + 1) \vdash n$ can be expressed as

$$\begin{aligned} c_{(n),i} + \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T \dim(M^{(\mu_1+1, \mu_2+1)})^{\mathfrak{S}_{(\mu'_1+1, \mu'_2+1)}} \\ = c_{(n),i} + \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T (\dim(M^\mu)^{\mathfrak{S}_{\mu'}} + 1) \end{aligned}$$

by Lemma 6.3. Now the above equation becomes:

$$\begin{aligned} c_{(n),i} + \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T + \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T \dim(M^\mu)^{\mathfrak{S}_{\mu'}} \\ = \dim(H^{2i}(\mathcal{H}ess(N, h))) + \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T \dim(M^\mu)^{\mathfrak{S}_{\mu'}} \end{aligned}$$

where in the last expression, both $\mu = (\mu_1, \mu_2)$ and $\mu' = (\mu'_1, \mu'_2)$ are partitions of $n - 2$, and the last equality follows from the case $\nu = (n)$ proven above. Since $\nu = (\mu'_1 + 1, \mu'_2 + 2)$, it follows from Proposition 6.6 that to prove (27) it is enough to prove

$$(29) \quad \dim H^{2i-2 \text{deg}(T)}(\mathcal{H}ess(X_{\mu'}, h_T)) = \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T \dim(M^\mu)^{\mathfrak{S}_{\mu'}}$$

for each $T \in \text{SK}_2(\Gamma_h)$. To see this, recall that the coefficients $c_{\mu, i - \text{deg}(T)}^T$ are defined by the equality

$$H^{2i-2 \text{deg}(T)}(\mathcal{H}ess(S', h_T)) = \sum_{\mu \vdash (n-2)} c_{\mu, i - \text{deg}(T)}^T M^\mu$$

in $\mathcal{R}ep(\mathfrak{S}_n)$. Moreover, as in the argument above, since I_h is abelian we know from Lemma 5.13 that $c_{\mu, i - \text{deg}(T)}^T = 0$ for all $\mu \vdash (n - 2)$ with more than 2 parts. This observation, together with taking $\mathfrak{S}_{\mu'}$ -invariants for $\mu' = (\mu'_1, \mu'_2) \vdash (n - 2)$, yields

$$(30) \quad \dim(H^{2i-2 \text{deg}(T)}(\mathcal{H}ess(S', h_T)))^{\mathfrak{S}_{\mu'}} = \sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T \dim(M^\mu)^{\mathfrak{S}_{\mu'}}.$$

Now another application of Theorem 2.16 on the LHS of (30) yields the equality in (29) as desired. Hence (27) holds for all ν with at most 2 parts, concluding the proof. \square

7. PROOFS OF PROPOSITIONS 6.5 AND 6.6 AND THE ABELIAN GRADED STANLEY–STEMBRIDGE CONJECTURE

In this section, we prove the two main inductive propositions from the previous section. The arguments involve the combinatorics of \mathfrak{S}_n and root systems. Given all of the preparation in the previous sections, the arguments are lengthy but not particularly difficult. *Throughout this section we work in the setting of Propositions 6.5 and 6.6 and Theorem 6.1. Thus we always assume that $n \geq 3$, that $h : [n] \rightarrow [n]$ is a Hessenberg function such that I_h is abelian, and that any partition has at most two parts.*

7.1. THE PROOF OF PROPOSITION 6.5. We begin with a proof of Proposition 6.5. This is much simpler than the proof of Proposition 6.6, which occupies the bulk of this section, due to the fact that the cohomology of the regular nilpotent Hessenberg variety $\mathcal{Hess}(\mathbf{N}, h)$ is related to the subspace of $H^*(\mathcal{Hess}(\mathbf{S}, h))$ which is invariant under the entire group \mathfrak{S}_n , as opposed to a Young subgroup \mathfrak{S}_ν for some $\nu \vdash n$. The fact that $\dim(M^\lambda)^{\mathfrak{S}_n} = 1$ for any partition λ then allows us to use Theorem 2.19 to translate our problem into the language of acyclic orientations. Our “sink-set decomposition” (13), and the inductive description of acyclic orientations given in Proposition 4.10, then yields the result. We now make this sketch precise.

Proof of Proposition 6.5. We begin by observing that

$$P(\mathcal{Hess}(\mathbf{N}, h), t) = \sum_{i=0}^{|\Phi_h^-|} \dim H^{2i}(\mathcal{Hess}(\mathbf{N}, h)) t^{2i} = \sum_{i=0}^{|\Phi_h^-|} \dim(H^{2i}(\mathcal{Hess}(\mathbf{S}, h)))^{\mathfrak{S}_n} t^{2i}$$

where the first equality is the definition of the Poincaré polynomial, together with the fact that $\dim_{\mathbb{C}}(\mathcal{Hess}(\mathbf{N}, h)) = |\Phi_h^-|$ [19, Corollary 2.7], and the second equality is by Brosnan and Chow’s Theorem 2.16. Since $H^{2i}(\mathcal{Hess}(\mathbf{S}, h)) = \sum_{\nu \vdash n} c_{\nu,i} M^\nu$ by definition of the coefficients $c_{\nu,i}$, by taking \mathfrak{S}_n -invariants we obtain

$$\begin{aligned} P(\mathcal{Hess}(\mathbf{N}, h), t) &= \sum_{i=0}^{|\Phi_h^-|} \left(\sum_{\nu \vdash n} c_{\nu,i} \dim(M^\nu)^{\mathfrak{S}_n} \right) t^{2i} \\ &= \sum_{i=0}^{|\Phi_h^-|} \left(\sum_{\nu \vdash n} c_{\nu,i} \right) t^{2i} \quad \text{since } \dim(M^\nu)^{\mathfrak{S}_n} = 1 \\ &= \sum_{i=0}^{|\Phi_h^-|} \left(\sum_{\substack{\nu \vdash n \text{ and } \nu \\ \text{has at most 2 parts}}} c_{\nu,i} \right) t^{2i} \quad \text{by Lemma 5.13} \\ &= \sum_{i=0}^{|\Phi_h^-|} c_{(n),i} t^{2i} + \sum_{i=0}^{|\Phi_h^-|} \left(\sum_{\substack{\nu \vdash n \text{ and } \nu \\ \text{has 2 parts}}} c_{\nu,i} \right) t^{2i}. \end{aligned}$$

A similar argument yields

$$P(\mathcal{Hess}(\mathbf{N}', h_T), t) = \sum_{i=0}^{|\Phi_{h_T}^-|} \left(\sum_{\mu \vdash (n-2)} c_{\mu,i}^T \right) t^{2i}$$

for any $T \in \text{SK}_2(\Gamma_h)$. The above equalities imply that in order to prove the proposition it suffices to prove

$$(31) \quad \sum_{i=0}^{|\Phi_h^-|} \left(\sum_{\substack{\nu \vdash n \text{ and } \nu \\ \text{has 2 parts}}} c_{\nu,i} \right) t^{2i} = \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} \sum_{i=0}^{|\Phi_{h_T}^-|} \left(\sum_{\mu \vdash (n-2)} c_{\mu,i}^T \right) t^{2i}.$$

Applying Theorem 2.19 and our previous combinatorial analysis of acyclic orientations we have

$$(32) \quad \sum_{i=0}^{|\Phi_h^-|} \left(\sum_{\substack{\nu \vdash n \text{ and } \nu \\ \text{has 2 parts}}} c_{\nu,i} \right) t^{2i} = \sum_{i=0}^{|\Phi_h^-|} |\{\omega \in \mathcal{A}_2(\Gamma_h) \mid \text{asc}(\omega) = i\}| t^{2i}$$

By our sink-set decomposition from (13) and an application of Proposition 4.10, the above equation becomes:

$$\begin{aligned}
 (33) \quad & \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{i=0}^{|\Phi_h^-|} |\{\omega \in \mathcal{A}_2(\Gamma_h) \mid \text{asc}(\omega) = i \text{ and } \text{sk}(\omega) = T\}| t^{2i} \\
 &= \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{i=\text{deg}(T)}^{|\Phi_h^-|} |\{\omega_T \in \mathcal{A}(\Gamma_{h_T}) \mid \text{asc}(\omega_T) = i - \text{deg}(T)\}| t^{2i}
 \end{aligned}$$

where the sum over the index i ranges between $\text{deg}(T)$ and $|\Phi_h^-|$ because it follows from Proposition 4.10 that if $\text{sk}(\omega) = T$ then $\text{asc}(\omega) \geq \text{deg}(T)$. For each $T \in \text{SK}_2(\Gamma_h)$ we shift the index i of the sum appearing on the RHS of (33) to get

$$\begin{aligned}
 (34) \quad & \sum_{i=\text{deg}(T)}^{|\Phi_h^-|} |\{\omega_T \in \mathcal{A}(\Gamma_{h_T}) \mid \text{asc}(\omega_T) = i - \text{deg}(T)\}| t^{2i} \\
 &= t^{2 \text{deg}(T)} \sum_{i=0}^{|\Phi_h^-| - \text{deg}(T)} |\{\omega_T \in \mathcal{A}(\Gamma_{h_T}) \mid \text{asc}(\omega_T) = i\}| t^{2i} \\
 &= t^{2 \text{deg}(T)} \sum_{i=0}^{|\Phi_{h_T}^-|} |\{\omega_T \in \mathcal{A}(\Gamma_{h_T}) \mid \text{asc}(\omega_T) = i\}| t^{2i}
 \end{aligned}$$

where the last equality follows from that fact that $|\Phi_h^-| - \text{deg}(T) \geq |\Phi_{h_T}^-|$ by Lemma 4.8 and $|\{\omega \in \mathcal{A}(\Gamma_{h_T}) \mid \text{asc}(\omega) = i\}| = 0$ for all $i > |\Phi_{h_T}^-|$ since $\text{asc}(\omega_T) \leq |E(\Gamma_{h_T})| = |\Phi_{h_T}^-|$ for all $\omega_T \in \mathcal{A}(\Gamma_{h_T})$. Putting together Corollary 2.21 with the above equation (34) we obtain

$$\begin{aligned}
 (35) \quad & \sum_{i=\text{deg}(T)}^{|\Phi_h^-|} |\{\omega_T \in \mathcal{A}(\Gamma_{h_T}) \mid \text{asc}(\omega_T) = i - \text{deg}(T)\}| t^{2i} \\
 &= t^{2 \text{deg}(T)} \sum_{i=0}^{|\Phi_{h_T}^-|} \left(\sum_{\mu \vdash (n-2)} c_{\mu,i}^T \right) t^{2i}
 \end{aligned}$$

for each $T \in \text{SK}_2(\Gamma_h)$. Finally, Equations (32), (33), and (35) together imply (31) as desired. \square

7.2. PROOF OF PROPOSITION 6.6. In this section we prove Proposition 6.6. This argument is the technical heart of this paper and is rather involved, so a sketch of the overall picture may be helpful. Our starting point is the explicit and purely combinatorial formula for the Betti numbers b_{2i} of the regular Hessenberg variety $\mathcal{H}_{\nu, h}$ given by the second author in [20] which expresses b_{2i} as the number of permutations $w \in \mathfrak{S}_n$ satisfying certain conditions related to $\nu \vdash n$ and h . Our assumptions that I_h is abelian and that all partitions have at most 2 parts simplifies the combinatorics of the Poincaré polynomial. From there, the remainder of the argument is a careful analysis of the sets of permutations in question, which boils down to the combinatorics of \mathfrak{S}_n and the root system of type A. There are two points worth mentioning. First, it turns out to be important that the formula for the Poincaré polynomial in [20] is valid

for any two-part composition $n = \nu_1 + \nu_2$ where $\nu = (\nu_1, \nu_2)$ is not necessarily a partition, i.e. we may have $\nu_1 < \nu_2$ instead of the more customary $\nu_1 \geq \nu_2$. Accordingly, in this section, the standing hypotheses on $\nu = (\nu_1, \nu_2)$ are as follows:

$$\nu_1, \nu_2 \in \mathbb{Z}, \quad \nu_1 + \nu_2 = n, \quad \nu_1 \geq 0, \nu_2 \geq 0.$$

Allowing this level of generality allows us to prove an important special case in our arguments below. Secondly, in order to reduce the argument to the special case mentioned in the previous sentence, we make use of a set of shortest coset representatives (also used by the second author in [20]) for the right cosets $W \backslash \mathfrak{S}_n$ where $W \subseteq \mathfrak{S}_n$ is a certain Young subgroup of \mathfrak{S}_n .

To begin, we state the formula for the Betti numbers of regular Hessenberg varieties given in [20]. We prepare some terminology. For each $w \in \mathfrak{S}_n$ we define the *inversion set of w* as

$$N^-(w) := \{\gamma \in \Phi^- \mid w(\gamma) \in \Phi^+\},$$

i.e. for each $w \in \mathfrak{S}_n$, the set $N^-(w)$ consists of the negative roots which become positive under the action of w . In Lie type A this can be expressed quite concretely. Indeed, let $\gamma = t_i - t_j$ for some $i > j$. The action of \mathfrak{S}_n on roots is given by

$$w(t_i - t_j) = t_{w(i)} - t_{w(j)}.$$

Thus $\gamma \in N^-(w)$ if and only if $w(i) < w(j)$. We may therefore naturally identify the set $N^-(w)$ with the set of ordered pairs

$$\text{inv}(w) := \{(i, j) \mid i > j \text{ and } w(i) < w(j)\}.$$

We use this identification frequently throughout this section.

We now state (a special case of) the key formula for the Poincaré polynomial of regular Hessenberg varieties [20, Lemma 2.6], which is the starting point of our discussion. Let $\nu = (\nu_1, \nu_2) \in \mathbb{Z}^2$ such that $\nu_1 + \nu_2 = n$ and $\nu_1, \nu_2 \geq 0$. There is a corresponding subset J_ν of the positive simple roots Δ given by

$$J_\nu := \Delta \setminus \{\alpha_\nu\} \subset \Delta \quad \text{where} \quad \alpha_\nu := t_{\nu_1} - t_{\nu_1+1}$$

if $0 < \nu_1 < n$ and $J_\nu = \Delta$ otherwise. Recall from Section 2.1 that $X_\nu = X_{(\nu_1, \nu_2)}$ denotes a matrix in standard Jordan canonical form which is regular of Jordan type ν . Recall also that the value of the Poincaré polynomial $P(\text{Hess}(X_\nu, h), t)$ when $t = 1$ is equal to $\dim H^*(\text{Hess}(X_\nu, h))$.

LEMMA 7.1 ([20, Lemma 2.6]). *Let n be a positive integer, $h : [n] \rightarrow [n]$ a Hessenberg function and $\nu = (\nu_1, \nu_2)$ as above. Then*

$$(36) \quad P(\text{Hess}(X_\nu, h), t) = \sum_{\substack{w \in \mathfrak{S}_n \\ w^{-1}(J_\nu) \subseteq \Phi_h^-}} t^{2|N^-(w) \cap \Phi_h^-|}.$$

In particular, evaluating at $t = 1$, we obtain

$$P(\text{Hess}(X_\nu, h), 1) = \dim H^*(\text{Hess}(X_\nu, h)) = |\{w \in \mathfrak{S}_n \mid w^{-1}(J_\nu) \subseteq \Phi_h\}|.$$

The exponents appearing in the RHS of (36) can be interpreted concretely in terms of the Hessenberg function. Indeed, under the identification of $N^-(w)$ with $\text{inv}(w)$, the set $N^-(w) \cap \Phi_h^-$ corresponds to

$$\text{inv}_h(w) := \{(i, j) \mid i > j, w(i) < w(j), \text{ and } i \leq h(j)\}.$$

Next, we analyze the indexing set of the summation on the RHS of (36), i.e. we study the set of $w \in \mathfrak{S}_n$ such that $w^{-1}(J_\nu) \subseteq \Phi_h$. Let $\nu = (\nu_1, \nu_2)$. Then $J_\nu = \Delta \setminus$

$\{\alpha_\nu\}$. Thus for any $w \in \mathfrak{S}_n$ with $w^{-1}(J_\nu) \subseteq \Phi_h$, we have that either $w^{-1}(\Delta) \subseteq \Phi_h$, or, $w^{-1}(J_\nu) \subseteq \Phi_h$ and $w^{-1}(\alpha_\nu) \in I_h$. Motivated by this observation we consider the set

$$\mathcal{D}_\nu := \{w \in \mathfrak{S}_n \mid w^{-1}(J_\nu) \subseteq \Phi_h \text{ and } w^{-1}(\alpha_\nu) \in I_h\}.$$

With this notation in place we obtain from Lemma 7.1 that

$$\begin{aligned} P(\mathcal{H}ess(X_\nu, h), t) &= \sum_{\substack{w \in \mathfrak{S}_n \\ w^{-1}(\Delta) \subseteq \Phi_h}} t^{2|N^-(w) \cap \Phi_h^-|} + \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} \\ (37) \qquad \qquad \qquad &= P(\mathcal{H}ess(\mathbb{N}, h), t) + \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} \end{aligned}$$

where the second equality follows from an application of Lemma 7.1 to the case $\nu = (n)$, the trivial composition with $\nu_1 = 0$ or $\nu_2 = 0$, and for which we may take $X_{(n)} = \mathbb{N}$ and $J_{(n)} = \Delta$. The discussion above indicates that the key step in the proof of Proposition 6.6 is the following.

PROPOSITION 7.2. *Under the notation and assumptions of Proposition 6.6, in particular with $\nu = (\mu_1 + 1, \mu_2 + 1) \vdash n$, we have*

$$(38) \qquad \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} = \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(X_\mu, h_T), t)$$

where $\mu = (\mu_1, \mu_2) \vdash (n - 2)$.

In order to prove Proposition 7.2, first recall from Proposition 5.8 that there is a bijection $\text{SK}_2(\Gamma_h) \rightarrow \mathcal{R}_1(I_h)$ which assigns each sink set T of cardinality 2 to a unique singleton set $R_T \in \mathcal{R}_1(I_h)$. Under this correspondence, if $R_T = \{\beta\}$, we write $\beta = \beta_T$. Moreover, by Proposition 5.6, for any Hessenberg function h , the ideal I_h bijectively corresponds to $\mathcal{R}_1(I_h)$ by $\beta \leftrightarrow \{\beta\}$. With this in mind, for each $\beta \in I_h$, we set the notation

$$\mathcal{D}_\nu(\beta) := \{w \in \mathfrak{S}_n \mid w^{-1}(J_\nu) \subseteq \Phi_h \text{ and } w^{-1}(\alpha_\nu) = \beta\}.$$

From the above discussion it is straightforward to see that

$$(39) \qquad \mathcal{D}_\nu = \bigsqcup_{\beta \in I_h} \mathcal{D}_\nu(\beta) = \bigsqcup_{T \in \text{SK}_2(\Gamma_h)} \mathcal{D}_\nu(\beta_T).$$

We can now rewrite (38) as

$$(40) \qquad \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{w \in \mathcal{D}_\nu(\beta_T)} t^{2|N^-(w) \cap \Phi_h^-|} = \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(X_\mu, h_T), t).$$

The proof of this equality consists of two parts. We first show an ungraded version of this equality (i.e. we prove that the equation above holds when we evaluate at $t = 1$) and then address the graded case. We need some additional notation for our proof.

Suppose $\beta = t_a - t_b \in I_h$. Recall that this is equivalent to the conditions $a > b$ and $a > h(b)$. If $w \in \mathcal{D}_\nu(\beta)$ then by definition of $\mathcal{D}_\nu(\beta)$ we must have

$$w^{-1}(t_{\nu_1} - t_{\nu_1+1}) = t_{w^{-1}(\nu_1)} - t_{w^{-1}(\nu_1+1)} = t_a - t_b,$$

or equivalently

$$(41) \qquad w(a) = \nu_1 \text{ and } w(b) = \nu_1 + 1,$$

so the b -th entry in the one-line notation of w is $\nu_1 + 1$ and the a -th entry is ν_1 .

In the arguments that follow it will be useful to choose a specific element of $\mathcal{D}_\nu(\beta)$ for each $\beta \in I_h$. We define this element as follows.

DEFINITION 7.3. *Suppose $\beta = t_a - t_b \in I_h$. We define a permutation in \mathfrak{S}_n , denoted $w_{\nu, \beta}$, by:*

- (1) $w_{\nu,\beta}(a) = \nu_1$ and $w_{\nu,\beta}(b) = \nu_1 + 1$ (i.e. $w_{\nu,\beta}$ satisfies the condition (41)) and
- (2) the remaining entries in the one-line notation of $w_{\nu,\beta}$ list the integers $[n] \setminus \{\nu_1, \nu_1 + 1\}$ in increasing order from left to right.

EXAMPLE 7.4. Let $n = 6$ and $\beta = t_5 - t_2$, so $a = 5$ and $b = 2$. Let $\nu = (4, 2)$. Then $w_{\nu,\beta}(2) = \nu_1 + 1 = 5$ and $w_{\nu,\beta}(5) = \nu_1 = 4$, and the remaining entries are filled, in increasing order, by $[6] \setminus \{4, 5\} = \{1, 2, 3, 6\}$. The one-line notation of $w_{\nu,\beta}$ is $[1 \mathbf{5} \mathbf{2} \mathbf{3} \mathbf{4} \mathbf{6}]$ where condition (41) determines the entries in bold.

We need the following.

LEMMA 7.5. *Let $w_{\nu,\beta}$ be as above and suppose I_h is abelian. Then*

- (1) *If $(i, j) \in \text{inv}(w_{\nu,\beta})$ then $\{i, j\} \cap \{a, b\} \neq \emptyset$, and*
- (2) *$w_{\nu,\beta} \in \mathcal{D}_\nu(\beta)$.*

Proof. To prove (1), we will show the contrapositive, i.e. if $\{i, j\} \cap \{a, b\} = \emptyset$ then $(i, j) \notin \text{inv}(w_{\nu,\beta})$. Suppose (i, j) is such that $i > j$ and $\{i, j\} \cap \{a, b\} = \emptyset$. Since $\{i, j\} \cap \{a, b\} = \emptyset$ we have $\{w_{\nu,\beta}(i), w_{\nu,\beta}(j)\} \cap \{\nu_1, \nu_1 + 1\} = \emptyset$, and it follows that $w_{\nu,\beta}(i) > w_{\nu,\beta}(j)$ by condition (2) in Definition 7.3. Therefore $(i, j) \notin \text{inv}(w_{\nu,\beta})$.

Now we prove (2). By definition, $w_{\nu,\beta}^{-1}(\alpha_\nu) = \beta$ so we need only show that $w_{\nu,\beta}^{-1}(J_\nu) \subseteq \Phi_h$. We take cases. First consider the case in which $\alpha \in J_\nu$ and $\alpha + \alpha_\nu \in \Phi$, i.e. α and α_ν correspond to adjacent vertices in the Dynkin diagram. Seeking a contradiction, suppose $w_{\nu,\beta}^{-1}(\alpha) \in I_h$. Since $\alpha + \alpha_\nu \in \Phi$ we also have $w_{\nu,\beta}^{-1}(\alpha + \alpha_\nu) = w_{\nu,\beta}^{-1}(\alpha) + w_{\nu,\beta}^{-1}(\alpha_\nu) \in I_h$ since I_h is an ideal. On the other hand, this is a contradiction since I_h is abelian. Next, consider the case in which $\alpha + \alpha_\nu \notin \Phi$, i.e. α and α_ν are not adjacent in the Dynkin diagram. This means that $\alpha = t_i - t_{i+1}$ where $\{i, i+1\} \cap \{\nu_1, \nu_1 + 1\} = \emptyset$. Condition (2) in Definition 7.3 implies $w_{\nu,\beta}^{-1}(i) < w_{\nu,\beta}^{-1}(i+1)$ since $i, i+1 \in [n] \setminus \{\nu_1, \nu_1 + 1\}$. Therefore $w_{\nu,\beta}^{-1}(\alpha) \in \Phi^+ \subseteq \Phi_h$ as desired. \square

With the element $w_{\nu,\beta}$ chosen as above, we can now construct some explicit maps which will be useful for our induction argument. Recall the bijection $\phi_T : [n] \setminus T \rightarrow [n - |T|]$ defined in Section 4.1 by $\phi_T(j) = j - j'$ where j' denotes the number of vertices $i \in T$ such that $i \leq j$. We now consider this bijection for the case in which $T = \{a, b\}$ more carefully. This bijection allows us to view $\mathfrak{gl}(n - 2, \mathbb{C})$ as a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. We make this identification more precise in equations (43), (44), and (45) below.

Viewing \mathfrak{S}_n as the automorphism group of the set $[n]$ of letters $\{1, 2, \dots, n\}$, there is a stabilizer subgroup

$$(42) \quad \text{Stab}(a, b) := \{w \in \mathfrak{S}_n \mid w(a) = a, w(b) = b\} \cong \text{Aut}([n] \setminus \{a, b\})$$

of \mathfrak{S}_n which is naturally isomorphic to \mathfrak{S}_{n-2} via the map

$$(43) \quad \tau \mapsto x_\tau := \left[\phi_T(\tau(1)) \phi_T(\tau(2)) \cdots \widehat{\phi_T(b)} \cdots \widehat{\phi_T(a)} \cdots \phi_T(\tau(n)) \right]$$

where the marked entries are deleted. In what follows we will frequently identify $\text{Stab}(a, b)$ with \mathfrak{S}_{n-2} . We have the following.

LEMMA 7.6. *Let $w \in \mathfrak{S}_n$ such that $w(a) = \nu_1$ and $w(b) = \nu_1 + 1$. Then there exists a unique $\tau \in \text{Stab}(a, b)$ such that $w = w_{\nu,\beta}\tau$. In particular, there is a well-defined map*

$$\Psi_{\nu,\beta} : \{w \in \mathfrak{S}_n \mid w(a) = \nu_1, w(b) = \nu_1 + 1\} \rightarrow \mathfrak{S}_{n-2}$$

defined by $\Psi_{\nu,\beta}(w) = x_\tau$ where x_τ is the unique element in \mathfrak{S}_{n-2} corresponding to $\tau \in \text{Stab}(a, b)$ via (43).

Sketch of proof. The hypotheses on w completely determine the a -th and b -th entries in its one-line notation. The other entries must be a permutation of the set $[n] \setminus \{a, b\}$, and the hypotheses on w place no condition on this permutation. Recall that for $w_{\nu, \beta}$ and any permutation τ in \mathfrak{S}_n , right-composition with τ “acts on positions”, i.e. if $w_{\nu, \beta}$ sends i to $w_{\nu, \beta}(i)$, then $w_{\nu, \beta}\tau$ must send i to $w_{\nu, \beta}(\tau(i))$. Thus if τ stabilizes a and b , then $w = w_{\nu, \beta}\tau$ still satisfies $w(a) = \nu_1$ and $w(b) = \nu_1 + 1$. Moreover, it is straightforward to see that such a τ is unique, making $\Psi_{\nu, \beta}$ well-defined. \square

EXAMPLE 7.7. Let $n = 5$, $\beta = t_5 - t_2$, and $\nu = (3, 2)$. Then $a = 5$, $b = 2$, and $\nu_1 = 3$. Let $h = (3, 4, 5, 5, 5)$, which is abelian, and I_h contains β . The table below gives an explicit description of the map $\Psi_{\nu, \beta}$ defined in Lemma 7.6.

$w \in \mathfrak{S}_5$ such that $w(5) = 3$ and $w(2) = 4$	$\tau \in \mathfrak{S}_5$	$x_\tau \in \mathfrak{S}_3$
$[1\ 4\ 2\ 5\ 3]$	$[1\ 2\ 3\ 4\ 5]$	$[1\ 2\ 3]$
$[2\ 4\ 1\ 5\ 3]$	$[3\ 2\ 1\ 4\ 5]$	$[2\ 1\ 3]$
$[1\ 4\ 5\ 2\ 3]$	$[1\ 2\ 4\ 3\ 5]$	$[1\ 3\ 2]$
$[2\ 4\ 5\ 1\ 3]$	$[3\ 2\ 4\ 1\ 5]$	$[2\ 3\ 1]$
$[5\ 4\ 1\ 2\ 3]$	$[4\ 2\ 1\ 3\ 5]$	$[3\ 1\ 2]$
$[5\ 4\ 2\ 1\ 3]$	$[4\ 2\ 3\ 1\ 5]$	$[3\ 2\ 1]$

There is a natural Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ obtained by “setting the variables in rows/columns a and b equal to zero”. More precisely, we have a natural Lie algebra isomorphism,

$$(44) \quad \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X_{ij} = 0 \text{ if } \{i, j\} \cap \{a, b\} \neq \emptyset\} \cong \mathfrak{gl}(n - 2, \mathbb{C})$$

defined explicitly on the basis $\{E_{ij} \mid \{i, j\} \cap \{a, b\} = \emptyset\}$ of the LHS by $E_{ij} \mapsto E_{\phi_T(i)\phi_T(j)}$ and extended linearly. Recall that in Section 4.1 we proved that each sink set $T \in \text{SK}_2(\Gamma_h)$ corresponds to a Hessenberg function $h_T : [n - 2] \rightarrow [n - 2]$ whose incomparability graph is obtained by deleting the vertices in T and any incident edges from Γ_h . In fact, this is the Hessenberg function which corresponds to the Hessenberg space $H \cap \mathfrak{gl}(n - 2, \mathbb{C})$ under the identification in (44), and ϕ_T can also be used to give an explicit map between the corresponding root systems. Using the notation of this section, $T = \{a, b\}$ so $\beta = \beta_T = t_a - t_b$. We let

$$\Phi[T] := \{t_i - t_j \mid 1 \leq i, j \leq n; \{i, j\} \cap \{a, b\} = \emptyset\} \subseteq \Phi$$

and Φ_T denote the root system of $\mathfrak{gl}(n - 2, \mathbb{C})$. Now there is an explicit isomorphism of root systems

$$(45) \quad \Phi[T] \cong \Phi_T \text{ defined by } t_i - t_j \mapsto t_{\phi_T(i)} - t_{\phi_T(j)}.$$

where $\Phi[T]$ is viewed as a subroot system of Φ (since $\Phi[T]$ is closed under addition in Φ). Moreover, if $\Phi^- [T] := \Phi^- \cap \Phi[T]$, $\Phi_h [T] := \Phi_h \cap \Phi[T]$, and $\Phi_h^- [T] := \Phi_h^- \cap \Phi[T]$ then these subsets of $\Phi[T]$ correspond to Φ_T^- , Φ_{h_T} , and $\Phi_{h_T}^-$ respectively via (45).

REMARK 7.8. The root system isomorphism given in (45) is compatible with the corresponding identification $\text{Stab}(a, b) \cong \mathfrak{S}_{n-2}$ given in (43). If $\tau \in \text{Stab}(a, b)$ and $t_i - t_j \in \Phi[T]$ then $\tau(t_i - t_j) \in \Phi[T]$ and

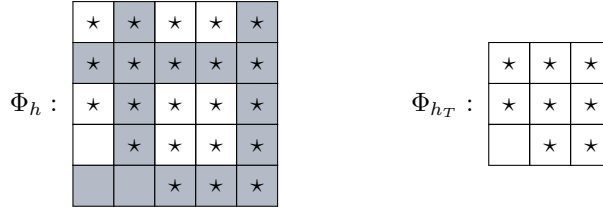
$$t_k - t_\ell = \tau(t_i - t_j) \Leftrightarrow t_{\phi_T(k)} - t_{\phi_T(\ell)} = x_\tau(t_{\phi_T(i)} - t_{\phi_T(j)}).$$

In particular, (45) maps $N^-(\tau)$ to $N^-(x_\tau)$.

EXAMPLE 7.9. Continuing Example 7.7, we get

$$\Phi^- [T] = \{t_3 - t_1, t_4 - t_1, t_4 - t_3\}$$

since $T = \{2, 5\}$. Pictorially, to obtain $\Phi[T]$ (respectively $\Phi^-[T]$) from Φ (respectively Φ^-) we simply remove those roots in the a -th and b -th rows and columns. The picture below illustrates the case $T = \{2, 5\}$.



LEMMA 7.10. *Let $T \in \text{SK}_2(\Gamma_h)$ and let $\beta = \beta_T$ be the corresponding element of I_h . Let $w_{\nu,\beta}$ be the permutation of Definition 7.3 associated to $\beta = \beta_T$.*

- (1) *If $\alpha_i = t_i - t_{i+1}$ is adjacent to α_ν in the Dynkin diagram (equivalently, $i = \nu_1 - 1$ or $i = \nu_1 + 1$), then $w_{\nu,\beta}^{-1}(\alpha_i) \notin \Phi[T]$.*
- (2) *If α_i is not adjacent to α_ν in the Dynkin diagram, then $w_{\nu,\beta}^{-1}(\alpha_i) \in \Phi_h[T]$.*

Furthermore, the set $w_{\nu,\beta}^{-1}(J_\nu) \cap \Phi[T]$ becomes the subset J_μ for $\mu = (\mu_1, \mu_2) \vdash (n - 2)$ via the identification in (45).

Proof. To prove (1), recall that $w_{\nu,\beta}^{-1}(\nu_1) = a$ and $w_{\nu,\beta}^{-1}(\nu_1 + 1) = b$ by definition. If $\alpha_i = t_i - t_{i+1}$ is adjacent to α_ν in the Dynkin diagram then

$$w_{\nu,\beta}^{-1}(\alpha_i) = t_{w_{\nu,\beta}^{-1}(i)} - t_{w_{\nu,\beta}^{-1}(i+1)} = \begin{cases} t_{w_{\nu,\beta}^{-1}(\nu_1-1)} - t_a & \text{if } i = \nu_1 - 1 \\ t_b - t_{w_{\nu,\beta}^{-1}(\nu_1+2)} & \text{if } i = \nu_1 + 1 \end{cases} .$$

In either case, $w_{\nu,\beta}^{-1}(\alpha) \notin \Phi[T]$.

Now we prove (2). If $\alpha_i = t_i - t_{i+1}$ is not adjacent to α_ν in the Dynkin diagram, then it is certainly the case that $w_{\nu,\beta}^{-1}(\alpha_i) \in \Phi[T]$ since $\{i, i + 1\} \cap \{\nu_1, \nu_1 + 1\} = \emptyset$ implies $\{w_{\nu,\beta}^{-1}(i), w_{\nu,\beta}^{-1}(i + 1)\} \cap \{a, b\} = \emptyset$. In addition $w_{\nu,\beta}^{-1}(\alpha_i) \in \Phi_h$ since $w_{\nu,\beta} \in \mathcal{D}_\nu(\beta)$ by Lemma 7.5.

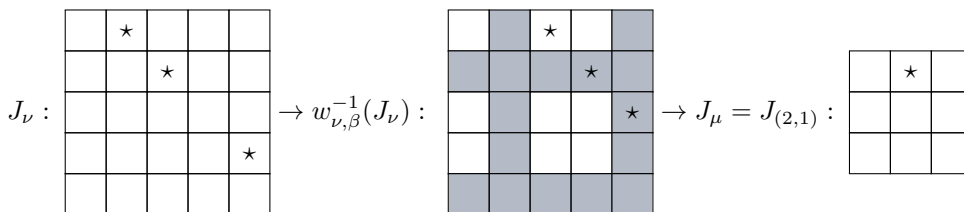
From (1) and (2) it follows that $\gamma \in w_{\nu,\beta}^{-1}(J_\nu) \cap \Phi[T]$ only if $\gamma = w_{\nu,\beta}^{-1}(\alpha)$ for some $\alpha = t_i - t_{i+1} \in J_\nu$ such that α is not adjacent to α_ν in the Dynkin diagram. In this case, $\{w_{\nu,\beta}^{-1}(i), w_{\nu,\beta}^{-1}(i + 1)\} \cap \{a, b\} = \emptyset$. Furthermore, since the entries in positions $n \setminus \{a, b\}$ of the one-line notation for $w_{\nu,\beta}$ increase from left to right it follows that $\phi_T(w_{\nu,\beta}^{-1}(i + 1)) = \phi_T(w_{\nu,\beta}^{-1}(i)) + 1$. In other words, $\alpha_{\phi_T(i)} = t_{\phi_T(i)} - t_{\phi_T(i+1)}$ is a positive simple root in Φ_T . Therefore the set $w_{\nu,\beta}^{-1}(J)$ for $J \subseteq \Delta$ defined by

- $J = \{\alpha_1, \dots, \alpha_{\nu_1-2}, \alpha_{\nu_1+2}, \dots, \alpha_{n-1}\}$ if $2 < \nu_1 < n - 2$, or
- $J = \{\alpha_3, \dots, \alpha_{n-1}\}$ if $\nu_1 = 1$, or
- $J = \{\alpha_4, \dots, \alpha_{n-1}\}$ if $\nu_1 = 2$, or
- $J = \{\alpha_1, \dots, \alpha_{n-4}\}$ if $\nu_1 = n - 2$, or
- $J = \{\alpha_1, \dots, \alpha_{n-3}\}$ if $\nu_1 = n - 1$

corresponds to a subset of positive simple roots in Φ_T . Finally, since (45) is an isomorphism of root systems, it preserves the addition of roots. Thus if two simple roots in J are adjacent in the Dynkin diagram for $\mathfrak{gl}(n, \mathbb{C})$, then their images under $w_{\nu,\beta}^{-1}$ and (45) are also adjacent in the Dynkin diagram for $\mathfrak{gl}(n - 2, \mathbb{C})$. It follows that $w_{\nu,\beta}^{-1}(J) = w_{\nu,\beta}^{-1}(J_\nu) \cap \Phi[T]$ is the subset J_μ via the identification given in (45). \square

EXAMPLE 7.11. Using the same set-up from Examples 7.7 and 7.9, we have $J_\nu = \{\alpha_1, \alpha_2, \alpha_4\}$ since $\nu = (3, 2)$. We track what happens to each of these simple roots under the action of $w_{\nu,\beta}^{-1}$ (where $w_{\nu,\beta} = [1 \ 4 \ 2 \ 5 \ 3]$) and subsequent identification

with simple roots in $\mathfrak{gl}(n - 2, \mathbb{C})$ below.



In addition, we consider the restriction of $\Psi_{\nu, \beta}$ to the set $\mathcal{D}_\nu(\beta)$. The table from Example 7.7 becomes

$w \in \mathcal{D}_\nu(\beta)$	$\tau \in \text{Stab}(a, b)$	$x_\tau \in \mathfrak{S}_3$
$[1\ 4\ 2\ 5\ 3]$	$[1\ 2\ 3\ 4\ 5]$	$[1\ 2\ 3]$
$[2\ 4\ 1\ 5\ 3]$	$[3\ 2\ 1\ 4\ 5]$	$[2\ 1\ 3]$
$[1\ 4\ 5\ 2\ 3]$	$[1\ 2\ 4\ 3\ 5]$	$[1\ 3\ 2]$
$[5\ 4\ 1\ 2\ 3]$	$[4\ 2\ 1\ 3\ 5]$	$[3\ 1\ 2]$
$[5\ 4\ 2\ 1\ 3]$	$[4\ 2\ 3\ 1\ 5]$	$[3\ 2\ 1]$

since $w = (2\ 4\ 5\ 1\ 3)$ does not satisfy the condition that $w^{-1}(J_\nu) \subseteq \Phi_h$. In addition, we note that the image of $\mathcal{D}_\nu(\beta)$ under $\Psi_{\nu, \beta}$ consists of those $x \in \mathfrak{S}_3$ such that $x^{-1}(J_\mu) \subseteq \Phi_{h_T}$. Our next lemma proves that this is true in general.

The next lemma identifies each $\mathcal{D}_\nu(\beta)$ with the set of permutations satisfying the ‘‘Hessenberg conditions’’ for $\text{Hess}(X_\mu, h_T)$, the smaller Hessenberg variety associated to h_T and $\mu \vdash (n - 2)$.

LEMMA 7.12. *Let $\tau \in \text{Stab}(a, b)$. The following are equivalent:*

- (1) $\tau^{-1}(w_{\nu, \beta}^{-1}(J_\nu)) \subseteq \Phi_h$ and
- (2) $\tau^{-1}(w_{\nu, \beta}^{-1}(J_\nu) \cap \Phi[T]) \subseteq \Phi_h[T]$.

In particular, the map $\Psi_{\nu, \beta}$ defined in Lemma 7.6 restricts to a bijection

$$\Psi_{\nu, \beta} : \mathcal{D}_\nu(\beta) \rightarrow \{x \in \mathfrak{S}_{n-2} \mid x^{-1}(J_\mu) \subseteq \Phi_{h_T}\}$$

where $\beta = \beta_T$.

Proof. The fact that (1) implies (2) is clear since τ preserves $\Phi[T]$ and $\Phi_h[T] = \Phi_h \cap \Phi[T]$. To prove that (2) implies (1), assume $\tau^{-1}(w_{\nu, \beta}^{-1}(J_\nu) \cap \Phi[T]) \subseteq \Phi_h[T]$ and let $\gamma \in \tau^{-1}(w_{\nu, \beta}^{-1}(J_\nu))$ so $\tau(\gamma) = w_{\nu, \beta}^{-1}(\alpha)$ for some $\alpha \in J_\nu$. Either α is adjacent to α_ν in the Dynkin diagram or it is not. We consider each case.

If α is not adjacent to α_ν in the Dynkin diagram, then statement (2) of Lemma 7.10 implies that $w_{\nu, \beta}^{-1}(\alpha) \in w_{\nu, \beta}^{-1}(J_\nu) \cap \Phi[T]$ so $\gamma = \tau^{-1}(w_{\nu, \beta}^{-1}(\alpha)) \in \tau^{-1}(w_{\nu, \beta}^{-1}(J_\nu) \cap \Phi[T]) \subseteq \Phi_h[T] \subseteq \Phi_h$. Now assume α is adjacent to α_ν in the Dynkin diagram, and for the sake of contradiction suppose that $\gamma = \tau^{-1}(w_{\nu, \beta}^{-1}(\alpha)) \in I_h$. Now $\alpha + \alpha_\nu \in \Phi$ and $(w_{\nu, \beta} \tau)^{-1}(\alpha), (w_{\nu, \beta} \tau)^{-1}(\alpha_\nu)$ are both elements of I_h . Their sum must also be an element of I_h , contradicting the assumption that I_h is abelian.

The last assertion of the Lemma now follows directly from the last assertion of Lemma 7.10, the identifications in (43), (45), and Remark 7.8. \square

We are now ready to prove an ungraded version of Proposition 7.2, i.e. for $t = 1$.

PROPOSITION 7.13. *Under the notation and assumptions of Proposition 6.6, let $T \in \text{SK}_2(\Gamma_h)$ and let $\beta = \beta_T$ as above. Then*

$$(46) \quad |\mathcal{D}_\nu(\beta)| = \dim H^*(\text{Hess}(X_\mu, h_T))$$

where $\mu = (\mu_1, \mu_2) \vdash (n - 2)$. In particular, the ungraded version of Proposition 7.2 holds, i.e.

$$|\mathcal{D}_\nu| = \sum_{T \in \text{SK}_2(\Gamma_h)} \dim H^*(\mathcal{H}ess(X_\mu, h_T)).$$

Proof. In the setting of the proposition, $\mu = (\mu_1, \mu_2)$ is a partition of $n - 2$. Recall from Lemma 7.1 that

$$(47) \quad \dim H^*(\mathcal{H}ess(X_\mu, h_T)) = |\{x \in \mathfrak{S}_{n-2} \mid x^{-1}(J_\mu) \subseteq \Phi_{h_T}\}|.$$

Thus to prove (46) it suffices to show that

$$(48) \quad |\mathcal{D}_\nu(\beta_T)| = |\{x \in \mathfrak{S}_{n-2} \mid x^{-1}(J_\mu) \subseteq \Phi_{h_T}\}|$$

but Lemma 7.12 establishes a bijective correspondence between the two sets in question, so (48) holds. Now the decomposition

$$\mathcal{D}_\nu = \bigsqcup_{T \in \text{SK}_2(\Gamma_h)} \mathcal{D}_\nu(\beta_T)$$

from (39) immediately yields

$$|\mathcal{D}_\nu| = \sum_{T \in \text{SK}_2(\Gamma_h)} |\mathcal{D}_\nu(\beta_T)| = \sum_{T \in \text{SK}_2(\Gamma_h)} \dim H^*(\mathcal{H}ess(X_\mu, h_T))$$

where the second equality uses (47) and (48) above. This concludes the proof. \square

We now turn our attention to the graded case, namely Proposition 7.2. The difficulty is that the inversion sets $N^-(w) \cap \Phi_h^- \cong \text{inv}_h(w)$ for $w \in \mathcal{D}_\nu(\beta)$ are not related to the inversion sets $N^-(\tau) \cap \Phi_h^-[T] \cong \text{inv}_{h_T}(x_\tau)$ by a simple formula. To remedy this, we need to shift each $w \in \mathcal{D}_\nu(\beta)$ by an appropriate translation, namely $w \mapsto \sigma_\nu w$ where σ_ν is the permutation defined below.

DEFINITION 7.14. Define $\sigma_\nu \in \mathfrak{S}_n$ by the following conditions:

- (1) $\sigma_\nu(\nu_1) = 1$ and $\sigma_\nu(\nu_1 + 1) = 2$, and
- (2) the remaining entries in the one-line notation of σ_ν list the integers $[n] \setminus \{1, 2\}$ in increasing order from left to right, i.e.

$$\sigma_\nu(i) = \begin{cases} i + 2 & \text{if } i < \nu_1 \\ i & \text{if } i > \nu_1 + 1 \end{cases}.$$

Note that σ_ν is uniquely determined by the value of ν_1 .

EXAMPLE 7.15. Using the same set-up as in Examples 7.7 and 7.9, recall that $n = 5$, $\nu = (3, 2)$, and $\beta = t_5 - t_2$. Since $\nu_1 = 3$ and $\nu_1 + 1 = 4$ we get

$$\sigma_\nu = [3 \mathbf{4} \mathbf{1} \mathbf{2} \mathbf{5}]$$

where condition (1) in Definition 7.14 determines the entries in bold and condition (2) determines the rest. Consider the translation $w \mapsto \sigma_\nu w$ for each $w \in \mathcal{D}_\nu(\beta)$, displayed in the table below.

$w \in \mathcal{D}_\nu(\beta)$	$\sigma_\nu w \in \mathfrak{S}_5$
[1 4 2 5 3]	[3 2 4 5 1]
[2 4 1 5 3]	[4 2 3 5 1]
[1 4 5 2 3]	[3 2 5 4 1]
[5 4 1 2 3]	[5 2 3 4 1]
[5 4 2 1 3]	[5 2 4 3 1]

Note that translation by σ_ν sends $4 \mapsto 2$ and $3 \mapsto 1$ in the one-line notation for w , but the rest of the entries of w remain in the same relative order in the one-line notation for $\sigma_\nu w$ as they were in the one-line notation of w .

The next lemma shows that translating the set $\mathcal{D}_\nu(\beta)$ by σ_ν does not change the inversions of w that are also elements of $\Phi^-[T]$.

LEMMA 7.16. *Suppose σ_ν is defined as above. For all $w \in \mathcal{D}_\nu(\beta)$ we have*

$$N^-(\sigma_\nu w) \cap \Phi^-[T] = N^-(w) \cap \Phi^-[T].$$

Proof. Recall that $\Phi^-[T] = \{t_i - t_j \in \Phi^- \mid \{i, j\} \cap \{a, b\} = \emptyset\}$. By definition, $w(a) = \nu_1$ and $w(b) = \nu_1 + 1$ so $\sigma_\nu w(a) = 1$ and $\sigma_\nu w(b) = 2$. Thus the a -th and b -th entry of $\sigma_\nu w$ in one-line notation is determined. By Condition (2) in Definition 7.14, σ_ν preserves the relative order of the values in the one-line notation of w which are not in positions a or b . It follows that for (i, j) with $\{i, j\} \cap \{a, b\} = \emptyset$ and $i > j$, we have $(i, j) \in \text{inv}(w)$ if and only if $(i, j) \in \text{inv}(\sigma_\nu w)$. \square

The next lemma relates the grading computation for $\sigma_\nu w$ to the grading computation for τ , up to a translation by $\text{deg}(T)$. This explains why it is useful to introduce the translation by σ_ν . One of the key points in the proof is that the LHS of (49) can be related to the edges of Γ_h which contribute to the computation of $\text{deg}(T)$.

LEMMA 7.17. *Let $w \in \mathcal{D}_\nu(\beta)$ for some $\beta = \beta_T \in I_h$ corresponding to $T \in \text{SK}_2(\Gamma_h)$. Let $w = w_{\nu, \beta \tau}$ be the decomposition of w given in Lemma 7.6 for a unique $\tau \in \text{Stab}(a, b)$. Then*

$$(49) \quad |N^-(\sigma_\nu w) \cap \Phi_h^-| = \text{deg}(T) + |N^-(\tau) \cap \Phi_h^-[T]|.$$

Proof. Since $\Phi^- = (\Phi^- \setminus \Phi^-[T]) \sqcup \Phi^-[T]$ we also have

$$\Phi_h^- = (\Phi_h^- \cap (\Phi^- \setminus \Phi^-[T])) \sqcup (\Phi_h^- \cap \Phi^-[T]).$$

Since $\Phi_h^- \cap \Phi^-[T]$ is the set $\Phi_h^-[T]$ by definition, we conclude

$$|N^-(\sigma_\nu w) \cap \Phi_h^-| = |N^-(\sigma_\nu w) \cap \Phi_h^- \cap (\Phi^- \setminus \Phi^-[T])| + |N^-(\sigma_\nu w) \cap \Phi_h^-[T]|.$$

Hence to prove (49) it suffices to prove that

$$(50) \quad |N^-(\sigma_\nu w) \cap \Phi_h^- \cap (\Phi^- \setminus \Phi^-[T])| = \text{deg}(T)$$

and

$$(51) \quad N^-(\sigma_\nu w) \cap \Phi_h^-[T] = N^-(\tau) \cap \Phi_h^-[T].$$

We first prove (50). By definition, $\Phi^- \setminus \Phi^-[T] = \{t_i - t_j \in \Phi^- \mid \{i, j\} \cap \{a, b\} \neq \emptyset\}$. Since $w \in \mathcal{D}_\nu(\beta)$, we know $w(a) = \nu_1$ and $w(b) = \nu_1 + 1$ by (41), and by construction of σ_ν this implies $\sigma_\nu w(a) = 1$ and $\sigma_\nu w(b) = 2$. It follows that 1 is in the a -th position of the one-line notation for $\sigma_\nu w$ and 2 is in the b -th position. Using the identification $N^-(\sigma_\nu w) \cong \text{inv}(\sigma_\nu w)$, we obtain

$$N^-(\sigma_\nu w) \cap (\Phi^- \setminus \Phi^-[T]) = \{(b, j) \mid 1 \leq j < b\} \cup \{(a, j) \mid 1 \leq j < a\}$$

and therefore

$$\begin{aligned} N^-(\sigma_\nu w) \cap \Phi_h^- \cap (\Phi^- \setminus \Phi^-[T]) \\ = \{(b, j) \mid 1 \leq j < b \text{ and } b \leq h(j)\} \cup \{(a, j) \mid 1 \leq j < a \text{ and } a \leq h(j)\}. \end{aligned}$$

Since $T = \{a, b\}$, the elements in the sets above correspond to edges of Γ_h that are incident to the vertices in T and which must be oriented to the right in order for a and b to be sinks. Thus, (50) now follows immediately from Lemma 4.8.

Next, in order to prove (51) we note that $N^-(\sigma_\nu w) \cap \Phi^-[T] = N^-(w) \cap \Phi^-[T]$ by Lemma 7.16. Intersecting both sides with Φ_h^- we obtain

$$(52) \quad N^-(\sigma_\nu w) \cap \Phi_h^-[T] = N^-(w) \cap \Phi_h^-[T].$$

Next we claim

$$(53) \quad N^-(w) \cap \Phi_h^-[T] = N^-(\tau) \cap \Phi_h^-[T].$$

As in the argument above, to see this it suffices to prove $N^-(w) \cap \Phi^-[T] = N^-(\tau) \cap \Phi^-[T]$, since (53) follows by intersecting both sides with Φ_h^- .

Suppose $t_i - t_j \in N^-(w) \cap \Phi^-[T]$. We wish to show $t_i - t_j \in N^-(\tau) \cap \Phi^-[T]$. By assumption we know $i > j$ and $\{i, j\} \cap \{a, b\} = \emptyset$ and $w_{\nu, \beta} \tau(i) = w(i) < w(j) = w_{\nu, \beta} \tau(j)$. Suppose in order to obtain a contradiction that $\tau(i) > \tau(j)$. Then $(\tau(i), \tau(j)) \in \text{inv}(w_{\nu, \beta})$ by the above. Moreover, since $\tau \in \text{Stab}(a, b)$ and $\{i, j\} \cap \{a, b\} = \emptyset$, we have $\{\tau(i), \tau(j)\} \cap \{a, b\} = \emptyset$ also. This contradicts part (1) of Lemma 7.5. Thus $\tau(i) < \tau(j)$, or equivalently $t_i - t_j \in N^-(\tau) \cap \Phi^-[T]$ as desired. Conversely, suppose $t_i - t_j \in N^-(\tau) \cap \Phi^-[T]$. Then $\tau(i) < \tau(j)$ and $\{\tau(i), \tau(j)\} \cap \{a, b\} = \emptyset$ and $w_{\nu, \beta} \tau(i) < w_{\nu, \beta} \tau(j)$ by condition (2) of Definition 7.3. Hence $t_i - t_j \in N^-(w) \cap \Phi^-[T]$. From the above it follows that $N^-(w) \cap \Phi^-[T] = N^-(\tau) \cap \Phi^-[T]$ and we obtain (53). Now (51) follows from (53) and (52). \square

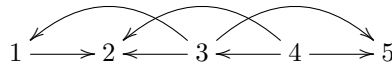
EXAMPLE 7.18. We confirm the results of Lemmas 7.16 and 7.17 for $n = 5, \nu = (3, 2)$, and $\beta = t_5 - t_2$. This is the case considered in Example 7.15, where $h = (3, 4, 5, 5, 5)$. The table below displays each $w \in \mathcal{D}_\nu(\beta)$ and computes $N^-(w) \cap \Phi_h^-[T]$ and $N^-(w) \cap \Phi_h^- \cap (\Phi^- \setminus \Phi^-[T])$. Here we use the identification $\text{inv}(w) \cong N^-(w)$.

$w \in \mathcal{D}_\nu(\beta)$	$N^-(w) \cap \Phi^-[T]$	$N^-(w) \cap \Phi_h^- \cap (\Phi^- \setminus \Phi^-[T])$
$[1 \ 4 \ 2 \ 5 \ 3]$	\emptyset	$\{(3, 2), (5, 4)\}$
$[2 \ 4 \ 1 \ 5 \ 3]$	$\{(3, 1)\}$	$\{(3, 2), (5, 4)\}$
$[1 \ 4 \ 5 \ 2 \ 3]$	$\{(4, 3)\}$	$\{(4, 2), (5, 3)\}$
$[5 \ 4 \ 1 \ 2 \ 3]$	$\{(3, 1), (4, 1)\}$	$\{(2, 1), (3, 2), (4, 2)\}$
$[5 \ 4 \ 2 \ 1 \ 3]$	$\{(3, 1), (4, 1), (4, 3)\}$	$\{(2, 1), (3, 2), (4, 2)\}$

Now we do the same computation for $\sigma_\nu w$.

$w \in \mathcal{D}_\nu(\beta)$	$\sigma_\nu w$	$N^-(\sigma_\nu w) \cap \Phi^-[T]$	$N^-(\sigma_\nu w) \cap \Phi_h^- \cap (\Phi^- \setminus \Phi^-[T])$
$[1 \ 4 \ 2 \ 5 \ 3]$	$[3 \ 2 \ 4 \ 5 \ 1]$	\emptyset	$\{(2, 1), (5, 3), (5, 4)\}$
$[2 \ 4 \ 1 \ 5 \ 3]$	$[4 \ 2 \ 3 \ 5 \ 1]$	$\{(3, 1)\}$	$\{(2, 1), (5, 3), (5, 4)\}$
$[1 \ 4 \ 5 \ 2 \ 3]$	$[3 \ 2 \ 5 \ 4 \ 1]$	$\{(4, 3)\}$	$\{(2, 1), (5, 3), (5, 4)\}$
$[5 \ 4 \ 1 \ 2 \ 3]$	$[5 \ 2 \ 3 \ 4 \ 1]$	$\{(3, 1), (4, 1)\}$	$\{(2, 1), (5, 3), (5, 4)\}$
$[5 \ 4 \ 2 \ 1 \ 3]$	$[3 \ 2 \ 4 \ 3 \ 1]$	$\{(3, 1), (4, 1), (4, 3)\}$	$\{(2, 1), (5, 3), (5, 4)\}$

The information in the tables above confirms the results of Lemma 7.16. The graph below shows the orientation ω of Γ_h with the property that $\text{sk}(\omega) = \{2, 5\}$ and $\text{asc}(\omega) = \text{deg}(T)$.



From the graph, we see $\text{deg}(T) = 3$, so the table above also confirms $|N^-(\sigma_\nu w) \cap \Phi_h^- \cap (\Phi^- \setminus \Phi^-[T])| = \text{deg}(T)$ for all $w \in \mathcal{D}_\nu(\beta)$. Finally, we consider each $\tau \in \text{Stab}(2, 5)$ such that $w = w_{\nu, \beta} \tau$ and compute both $N^-(\tau) \cap \Phi_h^-[T]$ and $N^-(\sigma_\nu w) \cap \Phi_h^-[T]$.

$w \in \mathcal{D}_\nu(\beta)$	$\tau \in \text{Stab}(2, 5)$	$N^-(\tau) \cap \Phi_h^-[T]$	$N^-(\sigma_\nu w) \cap \Phi_h^-[T]$
$[1 \ 4 \ 2 \ 5 \ 3]$	$[1 \ 2 \ 3 \ 4 \ 5]$	\emptyset	\emptyset
$[2 \ 4 \ 1 \ 5 \ 3]$	$[3 \ 2 \ 1 \ 4 \ 5]$	$\{(3, 1)\}$	$\{(3, 1)\}$
$[1 \ 4 \ 5 \ 2 \ 3]$	$[1 \ 2 \ 4 \ 3 \ 5]$	$\{(4, 3)\}$	$\{(4, 3)\}$
$[5 \ 4 \ 1 \ 2 \ 3]$	$[4 \ 2 \ 1 \ 3 \ 5]$	$\{(3, 1)\}$	$\{(3, 1)\}$
$[5 \ 4 \ 2 \ 1 \ 3]$	$[4 \ 2 \ 3 \ 1 \ 5]$	$\{(3, 1), (4, 3)\}$	$\{(3, 1), (4, 3)\}$

This table confirms that $N^-(w) \cap \Phi_h^-[T] = N^-(\sigma_\nu w) \cap \Phi_h^-[T] = N^-(\tau) \cap \Phi_h^-[T]$ as shown in the proof of Lemma 7.17.

COROLLARY 7.19. *Let $\nu = (\nu_1, \nu_2) = (\mu_1 + 1, \mu_2 + 1) \vdash n$ and σ_ν be defined as above. Then*

$$\sum_{\sigma_\nu w : w \in \mathcal{D}_\nu} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|} = \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(X_\mu, h_T), t).$$

Proof. Under the identifications in (43) and (45), τ becomes x_τ and $\Phi_h^-[T]$ becomes $\Phi_{h_T}^-$. By Remark 7.8 we have that $|N^-(\tau) \cap \Phi_h^-[T]| = |N^-(x_\tau) \cap \Phi_{h_T}^-|$. Therefore

$$\sum_{\sigma_\nu w : w \in \mathcal{D}_\nu} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|} = \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\sigma_\nu w : w \in \mathcal{D}_\nu(\beta_T)} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|}$$

since $\mathcal{D}_\nu = \bigsqcup_{T \in \text{SK}_2(\Gamma_h)} \mathcal{D}_\nu(\beta_T)$. We now obtain

$$\begin{aligned} \sum_{\sigma_\nu w : w \in \mathcal{D}_\nu} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|} &= \sum_{T \in \text{SK}_2(\Gamma_h)} \sum_{\substack{\sigma_\nu w_{\nu, \beta} \tau : \tau \in \text{Stab}(a, b) \\ x_\tau^{-1}(J_\mu) \subseteq \Phi_{h_T}}} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|} \\ &= \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} \sum_{\substack{x_\tau \in \mathfrak{S}_{n-2} \\ x_\tau^{-1}(J_\mu) \subseteq \Phi_{h_T}}} t^{2|N^-(x_\tau) \cap \Phi_{h_T}^-|} \end{aligned}$$

by applying Lemma 7.12 and Lemma 7.17. Finally, Lemma 7.1 implies that previous expression is equal to

$$\sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(X_\mu, h_T), t)$$

as desired. □

We now consider the special case $\nu = (n - 1, 1)$, which turns out to be critical. Note that $X_{(1, n-1)}$ and X_ν are conjugate. The basic trick in our proof below is to remember that the isomorphism class of Hessenberg varieties is preserved under conjugation, so in particular the Hessenberg varieties $\mathcal{H}ess(X_\nu, h)$ and $\mathcal{H}ess(X_{(1, n-1)}, h)$ are isomorphic and hence have the same Poincaré polynomial.

PROPOSITION 7.20. *Suppose $\nu = (n - 1, 1) \vdash n$. Then*

$$(54) \quad \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} = \sum_{\sigma_\nu w : w \in \mathcal{D}_\nu} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|}.$$

Proof. Consider the two partitions $\nu = (n - 1, 1)$ and $\nu' = (1, n - 1)$ of n . As already noted above, $\mathcal{H}ess(X_\nu, h) \cong \mathcal{H}ess(X_{\nu'}, h)$ and hence the two Hessenberg varieties have the same Poincaré polynomial. Notice that in both cases, the partition μ of $n - 2$ corresponding to the compositions defined by $\nu = (\nu_1, \nu_2) = (\mu_1 + 1, \mu_2 + 1)$ and $\nu' = (\nu'_1, \nu'_2) = (\mu_1 + 1, \mu_2 + 1)$ is the trivial partition of $n - 2$ and in particular the X_μ which appears in the RHS of both Proposition 7.2 and Corollary 7.19 for both cases $\nu = (n - 1, 1)$ and $\nu' = (1, n - 1)$, may be taken to be N' , the regular nilpotent element of $\mathfrak{gl}(n - 2, \mathbb{C})$. Also observe that $\sigma_{\nu'}$ is the identity permutation by definition, since $\nu'_1 = 1$ in this case.

From the above considerations we obtain:

$$\begin{aligned}
 P(\mathcal{H}ess(\mathbf{X}_\nu, h), t) &= P(\mathcal{H}ess(\mathbf{X}_{\nu'}, h), t) \\
 &= P(\mathcal{H}ess(\mathbf{N}, h), t) + \sum_{w : w \in \mathcal{D}_{\nu'}} t^{2|N^-(w) \cap \Phi_h^-|} \\
 &= P(\mathcal{H}ess(\mathbf{N}, h), t) + \sum_{\sigma_{\nu'} w : w \in \mathcal{D}_{\nu'}} t^{2|N^-(\sigma_{\nu'} w) \cap \Phi_h^-|} \\
 &= P(\mathcal{H}ess(\mathbf{N}, h), t) + \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(\mathbf{N}', h_T), t) \\
 &= P(\mathcal{H}ess(\mathbf{N}, h), t) + \sum_{\sigma_\nu w : w \in \mathcal{D}_\nu} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|}
 \end{aligned}$$

where the second equality is by Lemma 7.1 and (37) applied to $\nu' = (1, n - 1)$, the third follows from the fact that $\sigma_{\nu'} = e$, and the fourth (respectively fifth) equality is by applying Corollary 7.19 to $\nu' = (1, n - 1)$ (respectively $\nu = (n - 1, 1)$). On the other hand we also know

$$P(\mathcal{H}ess(\mathbf{X}_\nu, h), t) = P(\mathcal{H}ess(\mathbf{N}, h), t) + \sum_{w : w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|}$$

by Lemma 7.1 and (37) applied to $\nu = (n - 1, 1)$. Since the RHS of both of the above equalities must be equal, the equality in (54) follows. \square

The Proposition above proves a special case of our desired formula—namely, it implies that Proposition 7.2 holds for $\mathbf{X}_{(n-1,1)}$ when combined with Corollary 7.19. We now reduce to this special case using calculations involving the shortest coset representative.

Let $\nu = (\nu_1, \nu_2) \vdash n$. Consider the Young subgroup $W_\nu := \mathfrak{S}_{(\nu_1+1, 1, \dots, 1)}$ of \mathfrak{S}_n defined as permutations of $\{1, 2, \dots, \nu_1 + 1\} = [\nu_1 + 1]$. Note that W_ν is the Weyl group of $\mathfrak{gl}(\nu_1 + 1, \mathbb{C})$ viewed as a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ by identifying it with the upper left-hand $(\nu_1 + 1) \times (\nu_1 + 1)$ corner of the matrices in $\mathfrak{gl}(n, \mathbb{C})$. We will need the following for our proof of Proposition 7.2 below (see e.g. [15, Section 5]).

LEMMA 7.21. *Any $w \in \mathfrak{S}_n$ can be factored uniquely as $w = yz$ for some $y \in W_\nu$ and*

$$z \in {}^\nu W := \{v \in \mathfrak{S}_n \mid v^{-1}(\alpha_i) \in \Phi^+ \text{ for all } i = 1, \dots, \nu_1\}.$$

Moreover, $N^-(w) = N^-(z) \sqcup z^{-1}N^-(y)$.

The set ${}^\nu W$ is known as the *set of shortest coset representatives* for $W_\nu \backslash \mathfrak{S}_n$. The factors y and z in the decomposition given in Lemma 7.21 have a straightforward interpretation in terms of the one-line notation of w . More specifically, we can describe the one-line notation of z as follows. Suppose

$$w = [w(1) \ w(2) \ \cdots \ w(n)]$$

is the one-line notation of w . In order to obtain the one-line notation for z , we look at the entries in w which lie in $\{1, 2, \dots, \nu_1 + 1\}$, and re-write them in increasing order from left to right. All other entries remain unchanged. The result is the one-line notation for z .

EXAMPLE 7.22. For example, if $n = 7$ and $\nu_1 + 1 = 4$ and $w = [6 \ \mathbf{4} \ \mathbf{1} \ \mathbf{7} \ \mathbf{2} \ \mathbf{5} \ \mathbf{3}]$ where the numbers in boldface correspond to the entries in $\{1, 2, 3, 4\}$, by re-ordering just these entries we obtain $z = [6 \ \mathbf{1} \ \mathbf{2} \ \mathbf{7} \ \mathbf{3} \ \mathbf{5} \ \mathbf{4}]$. Now y is simply the element of $\mathfrak{S}_4 \subseteq \mathfrak{S}_7$ which permutes $\{1, 2, 3, 4\}$ to the ordering that was found in the original w , so in this

example $y = [4\ 1\ 2\ 3\ 5\ 6\ 7]$ which can also be viewed (since y stabilizes $\{5, 6, 7\}$) as $y = [4\ 1\ 2\ 3] \in \mathfrak{S}_4$.

EXAMPLE 7.23. Let $n = 5$ and $\nu = (3, 2)$ so $\nu_1 = 3$ and $\nu_1 + 1 = 4$ as in Example 7.7. Then $W_\nu = \mathfrak{S}_{\{1,2,3,4\}} \cong \mathfrak{S}_4$. The set of shortest coset representatives for $\mathfrak{S}_4 \setminus \mathfrak{S}_5$ consist of the permutations w for which the values of $\{1, 2, 3, 4\}$ appear in increasing order in the one-line notation of w . Thus there are 5 elements in ${}^\nu W$ in this case:

$${}^\nu W = \{[1\ 2\ 3\ 4\ 5], [1\ 2\ 3\ 5\ 4], [1\ 2\ 5\ 3\ 4], [1\ 5\ 2\ 3\ 4], [5\ 1\ 2\ 3\ 4]\}.$$

EXAMPLE 7.24. To illustrate the decomposition $N^-(w) = N^-(z) \sqcup z^{-1}N^-(y)$ we consider the example $w = (6, 4, 1, 7, 2, 5, 3), z = (6, 1, 2, 7, 3, 5, 4)$ as in Example 7.22. Then

$$N^-(z) = \{(2, 1), (3, 1), (5, 1), (6, 1), (7, 1), (5, 4), (6, 4), (7, 4), (7, 6)\}$$

and $N^-(y) = \{(2, 1), (3, 1), (4, 1)\}$ so

$$z^{-1}N^-(y) = \{(3, 2), (5, 2), (7, 2)\}.$$

The reader can check that $N^-(w) = N^-(z) \sqcup z^{-1}N^-(y)$.

The following result is a special case of [18, Proposition 5.2].

LEMMA 7.25. *Suppose $h : [n] \rightarrow [n]$ is a Hessenberg function and $H \subseteq \mathfrak{gl}(n, \mathbb{C})$ is the associated Hessenberg space. Given $z \in \mathfrak{S}_n$, let $\dot{z} \in \text{GL}(n, \mathbb{C})$ denote the corresponding permutation matrix. For every $z \in {}^\nu W$*

$$H_z := \dot{z}H\dot{z}^{-1} \cap \mathfrak{gl}(\nu_1 + 1, \mathbb{C})$$

is a Hessenberg space of $\mathfrak{gl}(\nu_1 + 1, \mathbb{C})$.

As for Lemma 7.21 above, there is a straightforward way to interpret the Hessenberg space H_z in terms of the one-line notation for $z \in {}^\nu W$. Recall from Definition 2.2 that H is spanned by $\{E_{ij} \mid i \leq h(j)\}$, which implies $\dot{z}H\dot{z}^{-1} \cap \mathfrak{gl}(\nu_1 + 1, \mathbb{C})$ is spanned by $\{E_{ij} \mid i, j \in [\nu_1 + 1] \text{ and } z^{-1}(i) \leq h(z^{-1}(j))\}$. Now let $h_z : [\nu_1 + 1] \rightarrow [\nu_1 + 1]$ denote the Hessenberg function corresponding to H_z . From the definition we obtain $\Phi_{h_z} = z\Phi_h \cap \Phi_\nu$ where $\Phi_\nu := \{t_i - t_j \mid 1 \leq i, j \leq \nu_1 + 1\} \subseteq \Phi$ is the root system of $\mathfrak{gl}(\nu_1 + 1, \mathbb{C})$ considered as a subroot system of Φ . Similarly, $I_{h_z} = zI_h \cap \Phi_\nu$ is the ideal of Φ_ν^- corresponding to h_z .

We are finally ready to prove Proposition 7.2. The main idea is to decompose the set \mathcal{D}_ν by subdividing the elements according to their shortest coset representative. This allows us to reduce to the case in which $y \in W_\nu$ and $J_{(\nu_1, \nu_2)} \cap \Phi_\nu = J_{(\nu_1, 1)}$, the special case from Proposition 7.20.

Proof of Proposition 7.2. We first claim that it suffices to prove that for all $\nu = (\nu_1, \nu_2) = (\mu_1 + 1, \mu_2 + 1) \vdash n$, we have

$$(55) \quad \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} = \sum_{\sigma_\nu w : w \in \mathcal{D}_\nu} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|}.$$

Indeed, given (55) it follows from Corollary 7.19 that

$$(56) \quad \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} = \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\text{Hess}(X_\mu, h_T), t)$$

which is the desired claim of Proposition 7.2. We now proceed to prove (55).

Given $w \in \mathcal{D}_\nu$, let $w = yz$ with $y \in W_\nu$ and $z \in {}^\nu W$ be the decomposition from Lemma 7.21 and let σ_ν be as above. Note that, by definition, $\sigma_\nu \in W_\nu$. It follows that

$\sigma_\nu w = (\sigma_\nu y)(z)$, where $\sigma_\nu y \in W_\nu$ and $z \in {}^\nu W$, is the decomposition of $\sigma_\nu w$ from Lemma 7.21 above. Thus

$$N^-(w) = N^-(z) \sqcup z^{-1}N^-(y)$$

and

$$N^-(\sigma_\nu w) = N^-(z) \sqcup z^{-1}N^-(\sigma_\nu y).$$

In particular,

$$|N^-(w) \cap \Phi_h^-| = |N^-(z) \cap \Phi_h^-| + |z^{-1}N^-(y) \cap \Phi_h^-|.$$

Next note that the action of z gives a bijective correspondence between $z^{-1}N^-(y) \cap \Phi_h^-$ and

$$N^-(y) \cap z\Phi_h^- = N^-(y) \cap z\Phi_h^- \cap \Phi_\nu = N^-(y) \cap \Phi_{h_z}^-$$

where the first equality holds since $N^-(y) \subseteq \Phi_\nu$ and the second equality is by definition of h_z . Thus

$$(57) \quad |N^-(w) \cap \Phi_h^-| = |N^-(z) \cap \Phi_h^-| + |N^-(y) \cap \Phi_{h_z}^-|.$$

Similarly $|N^-(\sigma_\nu w) \cap \Phi_h^-| = |N^-(z) \cap \Phi_h^-| + |N^-(\sigma_\nu y) \cap \Phi_{h_z}^-|$.

For each $z \in {}^\nu W$, define

$$\mathcal{D}_{\nu,z} := \{w \in \mathcal{D}_\nu \mid w = yz \text{ for some } y \in W_\nu\}.$$

Note that $\mathcal{D}_{\nu,z}$ may be empty for some z . However, Lemma 7.21 guarantees that

$$\mathcal{D}_\nu = \bigsqcup_{z \in {}^\nu W} \mathcal{D}_{\nu,z}.$$

Fix z such that $\mathcal{D}_{\nu,z} \neq \emptyset$ and let $y \in \mathfrak{S}_\nu$ such that $w = yz \in \mathcal{D}_{\nu,z}$. We have that $w^{-1}(J_\nu) \subseteq \Phi_h$ and $w^{-1}(\alpha_\nu) \in I_h$ if and only if $y^{-1}(J_\nu) \subseteq z\Phi_h$ and $y^{-1}(\alpha_\nu) \in zI_h$. Next we claim that these two conditions hold if and only if $y^{-1}(J_\nu \cap \Phi_\nu) \subseteq \Phi_{h_z}$ and $y^{-1}(\alpha_\nu) \in I_{h_z}$. The implication in the forward direction is straightforward since we can intersect both conditions with Φ_ν and $y \in \mathfrak{S}_\nu$ preserves Φ_ν . Thus it suffices to show the reverse implication. By the definition of Φ_{h_z} and I_{h_z} , it in fact suffices to show that $y^{-1}(J_\nu) \subseteq z\Phi_h$. Note that $J_\nu \setminus (J_\nu \cap \Phi_\nu) = \{\alpha_{\nu_1+1}, \dots, \alpha_{n-1}\}$. Since we already know $y^{-1}(J_\nu \cap \Phi_\nu) \subseteq z\Phi_h$, it is enough to show $y^{-1}(\alpha_s) \in z\Phi_h$ for $\nu_1 + 1 \leq s \leq n - 1$. We take cases. For s with $\nu_1 + 2 \leq s \leq n - 1$, the fact that $y \in \mathfrak{S}_\nu$ implies $y^{-1}(\alpha_s) = \alpha_s$, and now the assumption that $w = yz$ lies in \mathcal{D}_ν implies the desired result. For $s = \nu_1 + 1$, suppose for a contradiction that $y^{-1}(\alpha_{\nu_1+1}) \notin z\Phi_h$. Then by definition of the ideal I_h we have $y^{-1}(\alpha_{\nu_1+1}) \in zI_h$. On the other hand, by assumption $y^{-1}(\alpha_{\nu_1})$ also lies in zI_h . Since α_{ν_1} and α_{ν_1+1} are adjacent in the Dynkin diagram (equivalently, $\alpha_{\nu_1} + \alpha_{\nu_1+1} \in \Phi$) and since I_h is an ideal, we conclude that $y^{-1}(\alpha_{\nu_1}) + y^{-1}(\alpha_{\nu_1+1}) \in zI_h$, i.e. $w^{-1}(\alpha_{\nu_1}) + w^{-1}(\alpha_{\nu_1+1}) \in I_h$ where both $w^{-1}(\alpha_{\nu_1})$ and $w^{-1}(\alpha_{\nu_1+1})$ are elements of I_h , but this contradicts the fact that I_h is an abelian ideal. Hence we must have $y^{-1}(\alpha_{\nu_1+1}) \in z\Phi_h$, as desired.

In addition, $J_\nu \cap \Phi_\nu = J_{(\nu_1,1)}$ when viewed as a subset of simple roots in $\mathfrak{gl}(\nu_1 + 1, \mathbb{C})$. The above considerations allow us to conclude

$$(58) \quad \{y \in W_\nu \mid yz \in \mathcal{D}_{\nu,z}\} = \{y \in W_\nu \mid y^{-1}(J_{(\nu_1,1)}) \subseteq \Phi_{h_z} \text{ and } y^{-1}(\alpha_\nu) \in I_{h_z}\}$$

where the RHS is equal to the set $\mathcal{D}_{(\nu_1,1)} \subseteq W_\nu$ corresponding to the Hessenberg function $h_z : [\nu_1 + 1] \rightarrow [\nu_1 + 1]$. Finally, since the z action on the set Φ is an automorphism it is straightforward from the definitions to see that I_h abelian implies that I_{h_z} is abelian. Moreover, from its definition it follows that $\sigma_{(\nu_1,1)}$ can be identified with σ_ν via the inclusion $W_\nu \cong \mathfrak{S}_{\nu_1+1} \hookrightarrow \mathfrak{S}_n$, since the first part of each of the partitions $\nu = (\nu_1, \nu_2)$ and $(\nu_1, 1)$ is the same.

The arguments above imply that

$$\sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} = \sum_{z \in {}^\nu W} \sum_{yz \in \mathcal{D}_{\nu,z}} t^{2|N^-(z) \cap \Phi_h^-| + 2|N^-(y) \cap \Phi_{h_z}^-|}$$

using (57) and the decomposition $\mathcal{D}_\nu = \bigsqcup_{z \in {}^\nu W} \mathcal{D}_{\nu,z}$. We obtain

$$\begin{aligned} \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} &= \sum_{z \in {}^\nu W} t^{2|N^-(z) \cap \Phi_h^-|} \sum_{y \in \mathcal{D}_{(\nu_1,1)}} t^{2|N^-(y) \cap \Phi_{h_z}^-|} \\ &= \sum_{z \in {}^\nu W} t^{2|N^-(z) \cap \Phi_h^-|} \sum_{\sigma_{(\nu_1,1)}y : y \in \mathcal{D}_{(\nu_1,1)}} t^{2|N^-(\sigma_{(\nu_1,1)}y) \cap \Phi_{h_z}^-|} \end{aligned}$$

by applying (58) and Proposition 7.20 to the above. Finally, the fact that $\sigma_{(\nu_1,1)} = \sigma_\nu$ allows us to apply (57) to $\sigma_\nu w = (\sigma_\nu y)(z)$; the RHS of the above equation becomes

$$\begin{aligned} \sum_{z \in {}^\nu W} \sum_{\sigma_\nu yz : yz \in \mathcal{D}_{\nu,z}} t^{2|N^-(z) \cap \Phi_h^-| + 2|N^-(\sigma_\nu y) \cap \Phi_{h_z}^-|} &= \sum_{z \in {}^\nu W} \sum_{\sigma_\nu w : w \in \mathcal{D}_{\nu,z}} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|} \\ &= \sum_{\sigma_\nu w : w \in \mathcal{D}_\nu} t^{2|N^-(\sigma_\nu w) \cap \Phi_h^-|} \end{aligned}$$

proving (55), as desired. □

The proof of our main technical proposition is now a simple matter.

Proof of Proposition 6.6. As already noted in (37), we obtain

$$P(\mathcal{H}ess(\mathbf{X}_\nu, h), t) = P(\mathcal{H}ess(\mathbf{N}, h), t) + \sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|}$$

from Lemma 7.1. Now Proposition 7.2 says

$$\sum_{w \in \mathcal{D}_\nu} t^{2|N^-(w) \cap \Phi_h^-|} = \sum_{T \in \text{SK}_2(\Gamma_h)} t^{2 \deg(T)} P(\mathcal{H}ess(\mathbf{X}_\mu, h_T), t)$$

so the desired statement follows immediately. □

7.3. PROOF OF THE GRADED STANLEY–STEMBRIDGE CONJECTURE FOR THE ABELIAN CASE. We can now prove the graded Stanley–Stembridge conjecture for the abelian case by induction. We have the following.

COROLLARY 7.26. *Let n be a positive integer and $h : [n] \rightarrow [n]$ a Hessenberg function such that I_h is abelian. Then the integers $c_{\lambda,i}$ appearing in (9) are non-negative.*

Proof. We argue by induction. Our base cases are $n = 1$ and $n = 2$. The case $n = 1$ is trivial in the sense that the regular semisimple Hessenberg variety under consideration is just a single point, and the symmetric group is the trivial group. Hence the claim holds in this case.

The next case $n = 2$ is the first case in which the corresponding flag variety $\mathcal{F}lags(\mathbb{C}^n) = \mathcal{F}lags(\mathbb{C}^2) \cong \mathbb{P}^1$ is non-trivial. In this case there are only two Hessenberg functions to consider: $h = (1, 2)$ and $h = (2, 2)$. Both cases correspond to abelian ideals. If $h = (1, 2)$, the corresponding variety $\mathcal{H}ess(\mathbf{S}, (1, 2))$ consists of two points $\{N, S\}$ (the “north pole” and “south pole” of the \mathbb{P}^1) and hence its cohomology is non-zero only in degree 0. In this case, the corresponding Hessenberg space H is the Borel subalgebra and Teff’s results [28] prove that $H^0(\mathcal{H}ess(\mathbf{S}, (1, 2))) \cong M^{(1,1)}$. (The reader may confirm this by computing the corresponding representation directly using, for example, the explicit description of Tymoczko’s action in [30] via GKM theory.) If $h = (2, 2)$, the variety $\mathcal{H}ess(\mathbf{S}, (2, 2))$ is equal to the entire flag variety \mathbb{P}^1 and has non-zero cohomology only in degrees 0 and 2. Another direct computation shows

that $H^0(\mathcal{H}ess(\mathbf{S}, (2, 2))) \cong M^{(2)}$ and $H^2(\mathcal{H}ess(\mathbf{S}, (2, 2))) \cong M^{(2)}$. In both cases we conclude that the \mathfrak{S}_2 -representation, in each degree, is a non-negative sum of tabloid representations. Thus the claim of the theorem holds in these base cases.

Now suppose $n \geq 3$ and let \mathbf{S}' be any regular semisimple element of $\mathfrak{gl}(k, \mathbb{C})$ for k for $1 \leq k \leq n - 1$. Suppose also by induction that for any Hessenberg function $h' : [k] \rightarrow [k]$ with $I_{h'}$ abelian, we have that

$$H^{2i}(\mathcal{H}ess(\mathbf{S}', h')) = \sum_{\lambda \vdash k} c'_{\lambda,i} M^\lambda$$

where the $c'_{\lambda,i}$ are all non-negative. We wish to show that the corresponding statement is true for $k = n$. To see this, fix $i \geq 0$. Suppose $h : [n] \rightarrow [n]$ is a Hessenberg function such that I_h is abelian. We wish to show that

$$H^{2i}(\mathcal{H}ess(\mathbf{S}, h)) = \sum_{\lambda \vdash n} c_{\lambda,i} M^\lambda$$

where each $c_{\lambda,i} \in \mathbb{Z}$ and $c_{\lambda,i} \geq 0$.

On the other hand, by Theorem 6.1 we know

$$H^{2i}(\mathcal{H}ess(\mathbf{S}, h)) = c_{(n),i} M^{(n)} + \sum_{T \in \text{SK}_2(\Gamma_h)} \left(\sum_{\substack{\mu \vdash (n-2) \\ \mu = (\mu_1, \mu_2)}} c_{\mu, i - \text{deg}(T)}^T M^{(\mu_1+1, \mu_2+1)} \right).$$

On the RHS of the above equality, the coefficient $c_{(n),i}$ is non-negative by Corollary 2.20. Moreover, in the summation expression on the RHS, each μ is a partition of $n - 2$ and $c_{\mu, i - \text{deg}(T)}^T$ is the coefficient of M^μ in the decomposition of $H^{2i-2\text{deg}(T)}(\mathcal{H}ess(\mathbf{S}', h_T))$. Since h is an abelian Hessenberg function, h_T is also abelian by Lemma 5.13, and therefore each coefficient $c_{\mu, i - \text{deg}(T)}^T$ is non-negative by the induction hypothesis. Thus, the above equality expresses $H^{2i}(\mathcal{H}ess(\mathbf{S}, h))$ as a non-negative linear combination of tabloid representations. This completes the inductive step and hence the proof of the theorem. \square

8. A CONJECTURE FOR THE GENERAL CASE

Although we prove our main result, Theorem 1.1, for abelian Hessenberg varieties only, much of the framework and analysis in Sections 4 and 5 is general. In particular the analysis of maximal sink sets of the graph Γ_h in Section 4 shows that every acyclic orientation of Γ_h corresponding to such a sink set T is obtained inductively from an acyclic orientation of the smaller graph Γ_{h_T} on $n - |T|$ vertices. Using Theorem 2.19, this indicates a correspondence between the representations $H^*(\mathcal{H}ess(\mathbf{S}, h))$ and $H^*(\mathcal{H}ess(\mathbf{S}', h_T))$, where \mathbf{S}' denotes a regular semisimple element in $\mathfrak{gl}(n - |T|, h_T)$. Suppose $\lambda \vdash n$ has $m(\Gamma_h)$ parts. We now conjecture a formula for the coefficient of M^λ occurring in $H^*(\mathcal{H}ess(\mathbf{S}, h))$ as a function of the coefficients of M^μ occurring in $H^*(\mathcal{H}ess(\mathbf{S}', h_T))$ where $\mu \vdash (n - |T|)$.

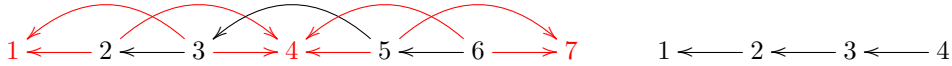
CONJECTURE 8.1. *Let $h : [n] \rightarrow [n]$ be a Hessenberg function and $\lambda \vdash n$ be a partition with exactly $m = m(\Gamma_h)$ parts. Let $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ be the partition of $n - |T|$ such that $\lambda = (\mu_1 + 1, \mu_2 + 1, \dots, \mu_m + 1)$. Then for all $i \geq 0$,*

$$c_{\lambda,i} = \sum_{T \in \text{SK}_m(\Gamma_h)} c_{\mu, i - \text{deg}(T)}^T.$$

This conjecture extends the results of Theorem 6.1 to arbitrary regular semisimple Hessenberg varieties. However, unless the Hessenberg variety is abelian (i.e. unless

$m(\Gamma_h) \leq 2$), this formula does not determine the entire representation $H^*(\mathcal{Hess}(\mathcal{S}, h))$. The next example demonstrates this.

EXAMPLE 8.2. Suppose $n = 7$ and $h = (3, 4, 5, 6, 7, 7, 7)$. In this case, $m(\Gamma_h) = 3$ so h is not an abelian. We will show that Conjecture 8.1 correctly predicts the coefficients of M^λ for $\lambda \vdash 7$ with exactly three parts. Note that, in this case, there is only one maximal sink set, namely $T = \{1, 4, 7\}$. The graph below shows the acyclic orientation $\omega \in \mathcal{A}_2(\Gamma_h)$ such that $\text{asc}(\omega) = \deg(\{1, 4, 7\})$. The vertices in $\{1, 4, 7\}$ and incident edges are highlighted in red, and we display the corresponding acyclic orientation of $\Gamma_h - T \cong \Gamma_{h_T}$ on the right.



The graphs above show that $\deg(\{1, 4, 7\}) = 4$ and $h_T = (2, 3, 4, 4)$. The table below computes the representation $H^*(\mathcal{Hess}(\mathcal{S}_T, h_T))$ as a sum of tabloid representations in each degree.

$H^0(\mathcal{Hess}(\mathcal{S}', h_T))$	$M^{(4)}$
$H^2(\mathcal{Hess}(\mathcal{S}', h_T))$	$M^{(4)} + M^{(3,1)} + M^{(2,2)}$
$H^4(\mathcal{Hess}(\mathcal{S}', h_T))$	$M^{(4)} + M^{(3,1)} + M^{(2,2)}$
$H^6(\mathcal{Hess}(\mathcal{S}', h_T))$	$M^{(4)}$

The next table shows the tabloid representations corresponding to partitions with 3 parts that occur as summands of $H^*(\mathcal{Hess}(\mathcal{S}, h))$ in $\mathcal{Rep}(\mathfrak{S}_7)$.

$H^8(\mathcal{Hess}(\mathcal{S}, h))$	$M^{(5,1,1)}$
$H^{10}(\mathcal{Hess}(\mathcal{S}, h))$	$M^{(5,1,1)} + M^{(4,2,1)} + M^{(3,3,1)}$
$H^{12}(\mathcal{Hess}(\mathcal{S}, h))$	$M^{(5,1,1)} + M^{(4,2,1)} + M^{(3,3,1)}$
$H^{14}(\mathcal{Hess}(\mathcal{S}, h))$	$M^{(5,1,1)}$

These tables confirm Conjecture 8.1. Although the conjectured formula correctly determines the tabloid representations M^λ appearing for λ with three parts, it does *not* determine the entire representation. For example,

$$\begin{aligned}
 &H^{10}(\mathcal{Hess}(\mathcal{S}, h)) \\
 &= 32M^{(7)} + 27M^{(6,1)} + 19M^{(5,2)} + 15M^{(4,3)} + M^{(5,1,1)} + M^{(4,2,1)} + M^{(3,3,1)}
 \end{aligned}$$

and we do not know of an inductive formula for the coefficients of $M^{(6,1)}$, $M^{(5,2)}$, or $M^{(4,3)}$ at this time.

The formula given in Conjecture 8.1 does not determine the coefficients for M^λ unless λ has a maximal number of parts. In this sense, this conjecture represents *the tip of an iceberg*. To obtain a formula which fully generalizes Theorem 6.1 we need some idea of how to inductively obtain the coefficients c_λ for λ with any number of parts. We intend to pursue this in future work.

Acknowledgements. The research for this manuscript was partially conducted at the Osaka City University Advanced Mathematical Institute, and we thank the Institute for its hospitality. We are also grateful to Timothy Chow, Seung Jin Lee, Mikiya Masuda, John Shareshian, Richard Stanley, and Stephanie van Willigenburg for valuable comments.

REFERENCES

- [1] Hiraku Abe, Megumi Harada, Tatsuya Horiguchi, and Mikiya Masuda, *The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A*, Int. Math. Res. Not. **2019** (2019), no. 17, 5316–5388.
- [2] Hiraku Abe, Tatsuya Horiguchi, and Mikiya Masuda, *The cohomology rings of regular semisimple Hessenberg varieties for $h = (h(1), n, \dots, n)$* , J. Comb. **10** (2019), no. 1, 27–59.
- [3] Patrick Brosnan and Timothy Y. Chow, *Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties*, Adv. Math. **329** (2018), 955–1001.
- [4] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli, *Representation theory of the symmetric groups*, Cambridge Studies in Advanced Mathematics, vol. 121, Cambridge University Press, Cambridge, 2010.
- [5] Soojin Cho and JiSun Huh, *On e -positivity and e -unimodality of chromatic quasisymmetric functions*, <https://arxiv.org/abs/1711.07152>, 2017.
- [6] Filippo De Mari, Claudio Procesi, and Mark A. Shayman, *Hessenberg varieties*, Trans. Amer. Math. Soc. **332** (1992), no. 2, 529–534.
- [7] Filippo De Mari Casareto Dal Verme, *On the topology of the Hessenberg varieties of a matrix*, Ph.D. thesis, Washington University in St. Louis (USA), 1987.
- [8] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.
- [9] Vesselin Gasharov, *Incomparability graphs of $(3 + 1)$ -free posets are s -positive*, Discrete Math. **157** (1996), no. 1-3, 193–197.
- [10] David D. Gebhard and Bruce E. Sagan, *A chromatic symmetric function in noncommuting variables*, J. Algebraic Combin. **13** (2001), no. 3, 227–255.
- [11] Mathieu Guay-Paquet, *A modular relation for the chromatic symmetric functions of $(3+1)$ -free posets*, <https://arxiv.org/abs/1306.2400>, 2013.
- [12] ———, *A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra*, <https://arxiv.org/abs/1601.05498>, 2016.
- [13] James Haglund, *The q, t -Catalan numbers and the space of diagonal harmonics*, University Lecture Series, vol. 41, American Mathematical Society, Providence, RI, 2008.
- [14] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1972.
- [15] Bertram Kostant, *Lie algebra cohomology and the generalized Borel–Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387.
- [16] ———, *The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations*, Int. Math. Res. Not. **1998** (1998), no. 5, 225–252.
- [17] Aba Mbirika and Julianna Tymoczko, *Generalizing Tanisaki’s ideal via ideals of truncated symmetric functions*, J. Algebraic Combin. **37** (2013), no. 1, 167–199.
- [18] Martha Precup, *Affine pavings of Hessenberg varieties for semisimple groups*, Selecta Math. (N.S.) **19** (2013), no. 4, 903–922.
- [19] ———, *The connectedness of Hessenberg varieties*, J. Algebra **437** (2015), 34–43.
- [20] ———, *The Betti numbers of regular Hessenberg varieties are palindromic*, Transform. Groups **23** (2018), no. 2, 491–499.
- [21] Md. Saidur Rahman, *Basic graph theory*, Undergraduate Topics in Computer Science, Springer, Cham, 2017.
- [22] John Shareshian and Michelle L. Wachs, *Chromatic quasisymmetric functions*, Adv. Math. **295** (2016), 497–551.
- [23] Eric Sommers and Julianna Tymoczko, *Exponents for B -stable ideals*, Trans. Amer. Math. Soc. **358** (2006), no. 8, 3493–3509 (electronic).
- [24] Richard P. Stanley, <http://front.math.ucdavis.edu/math.SG/0211231>, personal communication.
- [25] ———, *A symmetric function generalization of the chromatic polynomial of a graph*, Adv. Math. **111** (1995), no. 1, 166–194.
- [26] ———, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
- [27] Richard P. Stanley and John R. Stembridge, *On immanants of Jacobi–Trudi matrices and permutations with restricted position*, J. Combin. Theory Ser. A **62** (1993), no. 2, 261–279.
- [28] Nicholas James Teff, *The Hessenberg representation*, Ph.D. thesis, The University of Iowa (USA), 2013.
- [29] Julianna S. Tymoczko, *Linear conditions imposed on flag varieties*, Amer. J. Math. **128** (2006), no. 6, 1587–1604.

- [30] ———, *Permutation actions on equivariant cohomology of flag varieties*, in *Toric topology*, Contemp. Math., vol. 460, Amer. Math. Soc., Providence, RI, 2008, pp. 365–384.

MEGUMI HARADA, Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S4K1, Canada
E-mail : Megumi.Harada@math.mcmaster.ca
Url : <http://www.math.mcmaster.ca/Megumi.Harada/>

MARTHA E. PRECUP, Department of Mathematics and Statistics, Washington University in St. Louis, One Brookings Drive, St. Louis, Missouri 63130, U.S.A.
E-mail : martha.precup@wustl.edu
Url : <http://www.math.northwestern.edu/~mprecup/>