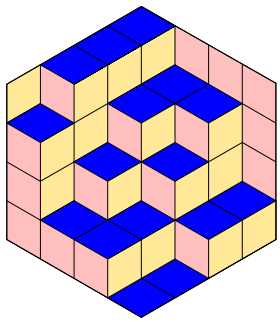

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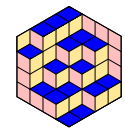

Promotion on oscillating and alternating tableaux and rotation of matchings and permutations

Stephan Pfannerer, Martin Rubey & Bruce Westbury

Abstract Using Henriques' and Kamnitzer's cactus groups, Schützenberger's promotion and evacuation operators on standard Young tableaux can be generalised in a very natural way to operators acting on highest weight words in tensor products of crystals.

For the crystals corresponding to the vector representations of the symplectic groups, we show that Sundaram's map to perfect matchings intertwines promotion and rotation of the associated chord diagrams, and evacuation and reversal. We also exhibit a map with similar features for the crystals corresponding to the adjoint representations of the general linear groups.

We prove these results by applying van Leeuwen's generalisation of Fomin's local rules for jeu de taquin, connected to the action of the cactus groups by Lenart, and variants of Fomin's growth diagrams for the Robinson–Schensted correspondence.

1. Introduction

This project began with the discovery that Sundaram's map from perfect matchings, regarded as chord diagrams as in Figure 1, to oscillating tableaux intertwines rotation and promotion, see Theorem 3.3. Oscillating tableaux are in bijection with highest weight words in a tensor power of the crystal of the vector representation of the symplectic group $\mathrm{Sp}(2n)$, and promotion is a natural generalisation of Schützenberger's promotion map on standard Young tableaux.

We then found a map analogous to Sundaram's from permutations, again regarded as chord diagrams, to Stembridge's alternating tableaux. Alternating tableaux of length r are in bijection with the highest weight words in the r-th tensor power of the crystal for the adjoint representation of the general linear group $\mathrm{GL}(n)$. This new map intertwines rotation and a suitable variant of promotion provided that $n \geqslant r$, see Theorem 3.7. Finally, it turned out that Theorem 3.3 can be deduced by a suitable embedding of the set of oscillating tableaux into the set of alternating tableaux.

Both results are part of a more elaborate program, as we now explain. A key observation is that Schützenberger's promotion and its above mentioned variants can be understood in terms of an action of the cactus groups. These infinite groups, also known as quasi-braid groups, were introduced by Devadoss [5, Def. 6.1.2] and placed into our context by Henriques and Kamnitzer [7]. They defined a weight preserving action of the r-fruit cactus group on highest weight words in r-fold tensor products

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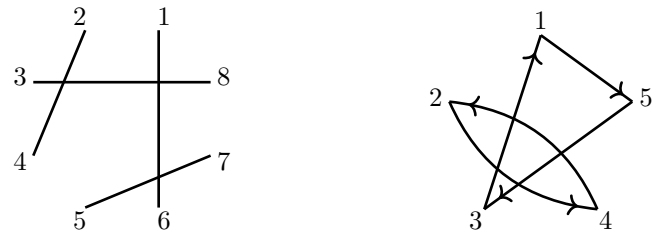


Figure 1. A 3-noncrossing perfect matching and a permutation as chord diagrams.

of crystals. The action of a specific element of the cactus group generalises promotion to an action on highest weight words in a tensor power of a crystal.

As shown by Lenart [11], the action of the cactus groups can be made explicit using certain local rules discovered by van Leeuwen [24]. These rules generalise the classical local rules for jeu de taquin by Fomin [21, App. 1] to crystals corresponding to any minuscule representation of a Lie group. To accommodate non-minuscule representations, one can use an embedding into tensor products of minuscule representations, with a few exceptions.

The link between highest weight words and diagrams involves classical invariant theory. Recall that the number of highest weight elements of given weight in a crystal is the multiplicity of the irreducible of the same weight in the direct sum decomposition of the corresponding representation. In particular, the number of highest weight elements of weight zero is the dimension of the invariant subspace.

The idea of using diagrams to index a basis of the invariant subspace of a tensor power of a representation goes back to Rumer, Teller and Weyl [19], and specifically Brauer [1]. Given a perfect matching of $2r$ elements, he constructed an invariant of the $2r$-th tensor power of the vector representation of the symplectic group $\mathrm{Sp}(2n)$. Furthermore, he showed that these invariants linearly span the invariant space. Using a result of Sundaram [23], Rubey and Westbury [17, 18] have shown that the invariants obtained from perfect matchings without $(n+1)$-crossings (see Section 3.1 for the definition) form a basis of this space.

The symmetric group acts on a tensor power of a representation by permuting tensor positions. This action commutes with the action of the Lie group. In particular, the symmetric group also acts on the invariant space of the tensor power.

It is not hard to see that Brauer's construction translates rotation of the chord diagram to the action of the long cycle of the symmetric group on the corresponding invariant. Moreover, it was shown by Westbury [25] that the action of the long cycle is isomorphic to the action of promotion on highest weight words of weight zero.

In general, given a representation, we would like to find a basis of the invariant space indexed by diagrams, such that rotation of diagrams corresponds to the action of the long cycle on the invariant space. We provide a review of such diagrammatic bases for invariant spaces of tensor powers of other representations in Section 2.

It then remains to establish an explicit bijection between the set of diagrams and the set of highest weight words of weight zero which intertwines rotation and promotion. This is the focus of the present article.

The initial motivation to study this problem arises from Reiner, Stanton and White's cyclic sieving phenomenon [15]. Essentially, this phenomenon occurs when the character of a cyclic group action can be expressed as a polynomial in a particularly simple way. A standard example is the rotation action on noncrossing perfect

matchings of $\{1,\dots,2r\}$. In this case, the q-Catalan 'number' $\frac{1}{[r+1]_q}\begin{bmatrix}2r\\ r\end{bmatrix}_q$ is a cyclic sieving polynomial: the evaluation at $q = e^{k\pi i/r}$ yields the number of noncrossing perfect matchings invariant under rotation by k points.

Consider the representation of the symmetric group on the invariant space, and recall that the diagrammatic basis is preserved by the action of the long cycle. Then the cyclic sieving polynomial for this cyclic action can be extracted from the Frobenius character of the symmetric group action. Although the Frobenius character of the invariant spaces of tensor powers of representations is in general hard to compute, it is known for several representations of interest, in particular for the vector representation of $\mathrm{Sp}(2n)$ and the adjoint representation of $\mathrm{GL}(n)$.

This article can therefore be regarded as an explicit demonstration of the fact that the cyclic sieving phenomena for promotion of highest weight words of weight zero and rotation of diagrams are the same in the case of the vector representation of $\mathrm{Sp}(2n)$ and in the case of the adjoint representation of $\mathrm{GL}(n)$.

In Section 2 we recall some background material on crystals of minuscule representations, make our goal precise and indicate the conditions necessary to make our methods work. Furthermore, we provide a summary of our contributions and what is already known. Precise statements of the new results are given in Section 3. The general machinery connecting the action of the cactus groups on highest weight words, local rules, and promotion is developed in Section 4. In Section 5 we introduce the two fundamental growth diagram bijections, which apply only to oscillating and alternating tableaux, respectively. The proofs that these bijections indeed intertwine promotion and rotation are delivered in the final section, along with some additional material.

2. Crystals and highest weight words

In this section, before stating the main goal of this article precisely, we provide some background information on minuscule representations, crystals and their tensor products. We also recall explicit combinatorial realisations of the associated highest weight words, and survey in which cases partial solutions to the questions mentioned in the introduction are known. Although we will subsequently only consider the adjoint representation of $\mathrm{GL}(n)$ and the vector representation of $\mathrm{Sp}(2n)$, we provide the background in a more general setting, to place our results into a bigger picture.

A representation of a Lie group is *minuscule* if its Weyl group W acts transitively on the weights of the representation: the set of weights forms a single orbit under the action of W. The non-trivial minuscule representations are:

Type A_n. All exterior powers of the vector representation.
Type B_n. The spin representation.
Type C_n. The vector representation.
Type D_n. The vector representation and the two half-spin representations.
Type E_6. The two fundamental representations of dimension 27.
Type E_7. The fundamental representation of dimension 56.

There are no nontrivial minuscule representations in types G_2, F_4 or E_8. Except for these types, any representation can be embedded into a tensor product of minuscule representations.

For example, the adjoint representation of $\mathrm{GL}(n)$ is not minuscule, but can be regarded as the tensor product of the vector representation and its dual. Slightly less trivial, the vector representation of the odd orthogonal group is not minuscule, but appears as a direct summand in $S \otimes S$, where S is the spin representation of the spin group $\mathrm{Spin}(2n+1)$.

Given a dominant weight λ, we associate to the irreducible representation $V(\lambda)$ its *crystal graph* B_λ. This is a certain connected edge-coloured digraph with $\dim V(\lambda)$ vertices, each labelled with a weight of the representation. Each edge of the crystal graph is labelled with one of the simple roots of the root system, such that the weight of the target of the edge is obtained from the weight of its source by subtracting the simple root. There is a unique vertex without in-coming edges, the *highest weight vertex*, and this vertex has weight λ. There is also a unique vertex without out-going edges, the *lowest weight vertex*. The sum of the formal exponentials of the weights of the vertices is the character of the representation. In particular, isomorphism of crystal graphs corresponds to isomorphism of representations. The direct sum of representations is then associated with the disjoint union of the corresponding crystal graphs.

For a minuscule representation $V(\lambda)$ of dominant weight λ, the vertices of the associated crystal graph B_λ can be identified with the weights of $V(\lambda)$. The edges are given by the Kashiwara lowering operators, as follows. Let $\{\alpha_i : i \in I\}$ be the set of simple roots and $s_i \in W$ be the simple reflection corresponding to α_i. Then there is a coloured edge $\mu \xrightarrow{i} \mu - \alpha_i$ provided that $s_i(\mu) = \mu - \alpha_i$.

There is a (relatively) simple way to construct the crystal graph of a tensor product of representations given their individual crystal graphs. The vertices of the tensor product $C_1 \otimes \cdots \otimes C_r$ of crystal graphs, corresponding to an r-fold tensor product of representations, are the words of length r whose i-th letter is a vertex of C_i. The weight of a vertex in the tensor product is the sum of the weights of its letters. In this context, we refer to the highest weight vertices of the connected components as *highest weight words*. Isolated vertices correspond to copies of the trivial representation and therefore have weight zero. They are referred to as *highest weight words of weight zero*.

There is an action of the so-called r-fruit cactus group $\mathfrak{C}_r$ on the set of highest weight words in r-fold tensor products of crystals $C_1, \ldots, C_r$. The cactus group $\mathfrak{C}_r$ has generators $\mathrm{s}_{p,q}$, for $1 \leqslant p < q \leqslant r$, satisfying certain relations stated in Definition 4.1 of Section 4. The generator $\mathrm{s}_{p,q}$ acts by mapping highest weight words of $C_1 \otimes \cdots \otimes C_r$ bijectively to highest weight words of

$$C_1 \otimes \cdots \otimes C_{p-1} \otimes (C_q \otimes C_{q-1} \cdots \otimes C_p) \otimes C_{q+1} \otimes \cdots \otimes C_r,$$

see Definition 4.5. We define the *promotion* $\operatorname{pr} w$ of w as $\mathrm{s}_{1,r}\,\mathrm{s}_{2,r}\,w$, and the *evacuation* $\operatorname{ev} w$ of w as $\mathrm{s}_{1,r}\,w$. These generalise Schützenberger's maps of the same name, as we explain in example 2.1 below. We will see in Lemma 4.2 that the cactus group $\mathfrak{C}_r$ is already generated by the elements $\mathrm{s}_{1,q}$ for $q \leqslant r$. Therefore it is essentially enough to understand evacuation. An analogous statement is true for promotion.

The main problem. Define a set of chord diagrams and a bijection between these and the highest weight words of weight zero which intertwines rotation of diagrams with promotion of words. Additionally, determine the action of evacuation on the set of chord diagrams.

In the following we briefly survey the cases in which (partial) solutions to this problem are known.

Recall that dominant weights of $\mathrm{SL}(n)$, $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$ are vectors of length n with weakly decreasing non-negative integer entries. Therefore, dominant weights can be identified with integer partitions into at most n parts in these cases. A dominant weight of $\mathrm{Spin}(2n+1)$ is a vector of length n with weakly decreasing non-negative half-integer entries, such that either all entries are integers or none of them. Finally, a dominant weight of $\mathrm{GL}(n)$ is a vector of length n with weakly decreasing integer entries, a so-called staircase.

Throughout, we denote the i-th unit vector by e_i. To improve readability, we will use vector notation for weights - often dropping commas and parentheses, and specify letters of highest weight words as linear combinations of unit vectors e_i.

To any highest weight word $w = w_1 \dots w_r$ in a tensor product of crystals $C_1 \otimes \cdots \otimes C_r$ we bijectively associate a sequence of dominant weights going under names like semistandard, oscillating, alternating, vacillating *tableau*. We call the final weight $\mu = (\mu_1, \dots, \mu_n)$ of such a sequence, which will also be the weight of the word w, the *shape* of the tableau. If μ is zero, we say that the tableau is of *empty shape*. We denote the zero weight by $\varnothing$.

Suppose now that in each crystal C_i, $1 \leqslant i \leqslant r$, all vertices have distinct weight. For example, this is the case when all the C_i correspond to minuscule representations. Then the tableau is given by the sequence $\varnothing = \mu^0, \mu^1, \dots, \mu^r = \mu$, where $\mu^q = \sum_{i=1}^{q} \mathrm{wt}(w_i)$ is the sum of the weights of the first q letters. In this case, one can recover the letters of the highest weight word via the successive differences $\mathrm{wt}(w_i) = \mu^i - \mu^{i-1}$.

In the examples below, the only exception to this rule is the case of alternating tableaux for the adjoint representation of $\mathrm{GL}(n)$, which we will explain separately.

Example 2.1 (Standard and semistandard tableaux). Let V be the vector representation of $\mathrm{SL}(n)$, and let each C_i, for $1 \leqslant i \leqslant r$, be a copy of the corresponding crystal. Then the highest weight words can be identified with standard Young tableaux of size r with at most n columns: the position of the unique entry equal to 1 in w_i is the column of the tableau in which the number i appears. Since the weight lattice of $\mathrm{SL}(n)$ is the image of $\mathbb{Z}^n$ in the quotient of $\mathbb{R}^n$ by the span of $(1, \dots, 1)$, a highest weight word has weight zero if and only if all n columns of the corresponding tableau have the same length.

More generally, for any sequence of positive integers $\alpha = \alpha_1, \dots \alpha_r$, let C_i be the crystal corresponding to the α_i-th exterior power of the vector representation of $\mathrm{SL}(n)$. Then w is a highest weight word if and only if w_i has exactly α_i entries equal to 1, all others 0, and μ^q is dominant for all $q \leqslant r$. Therefore, a highest weight word can be identified with a semistandard Young tableau of type α having at most n columns.[(1)]

One can show that in this case the generator $\mathrm{s}_{1,r}$ of the cactus group is precisely Schützenberger's evacuation of semistandard Young tableaux with largest entry at most r, and $\mathrm{s}_{1,r}\,\mathrm{s}_{2,r}$ is Schützenberger's promotion. Using evacuation as a building block, the action of the cactus groups on semistandard Young tableaux was studied by Chmutov, Glick and Pylyavskyy [3]. As an aside, we remark that the generators $\mathrm{s}_{i,i+2}$ encode Assaf's dual equivalence graph.

A diagrammatic basis for the invariant space was recently constructed by Cautis, Kamnitzer and Morrison [2], generalising Kuperberg's webs for SL(2) and SL(3), see [10]. However, only Kuperberg's web bases are preserved by rotation. For these, Petersen, Pylyavskyy and Rhoades [14] and Patrias [13] demonstrated that the growth algorithm of Khovanov and Kuperberg in [9] intertwines promotion with rotation.

Example 2.2 (Oscillating tableaux). Let V be the vector representation of $\mathrm{Sp}(2n)$ and let C_i be the corresponding crystal, for $1 \leqslant i \leqslant r$. Then w is a highest weight word if and only if w_i is in $\{\pm e_j : 1 \leqslant j \leqslant n\}$, and μ^q is dominant for $q \leqslant r$. The corresponding tableau is called an n-symplectic oscillating tableau.

For example, the 1-symplectic oscillating tableaux of length three are

$$(\varnothing, 1, 2, 3), \quad (\varnothing, 1, 2, 1), \quad \text{and} \quad (\varnothing, 1, \varnothing, 1).$$

[(1)] We are using slightly a nonstandard convention here. More traditionally, one would use *dual* semistandard tableaux instead, with entries in *columns weakly increasing*, entries in *rows strictly increasing*, and at most n *rows*. The positions containing a 1 in w_i then designate the *rows* of the tableau containing the number i.

The corresponding highest weight words are

$$e_1\, e_1\, e_1, \quad e_1\, e_1\, \text{-}e_1, \quad \text{and} \quad e_1\, \text{-}e_1\, e_1.$$

As a further example, the oscillating tableau

$$\mathcal{O} = (\varnothing, 1, 11, 21, 2, 21, 11, 21, 211, 21)$$

has length 9 and shape 21. It is 3-symplectic (since no partition has four parts) but it is not 2-symplectic (since there is a partition with three parts). The corresponding highest weight word is $w = e_1\, e_2\, e_1\, \text{-}e_2\, e_2\, \text{-}e_1\, e_1\, e_3\, \text{-}e_3$.

A suitable set of chord diagrams is the set of $(n+1)$-noncrossing perfect matchings of $\{1, \dots, r\}$, see Section 3.1 for definitions. A surjection from the set of perfect matchings to a basis of the invariant subspace of $\otimes^r V$ was given by Brauer [1]. Sundaram [23] provided a bijection between the set of $(n+1)$-noncrossing perfect matchings and n-symplectic oscillating tableaux of empty shape. Theorem 3.3 below shows that this bijection intertwines rotation with promotion, and reversal with evacuation.

A variation of this example is obtained by replacing V with the k-th symmetric power of the vector representation. A suitable set of chord diagrams indexing a basis of the space of invariant tensors was defined by Rubey and Westbury [18, 17]. Briefly, partition the set $\{1, \dots, kr\}$ into r blocks of k consecutive elements. Then a chord diagram is an $(n+1)$-noncrossing perfect matching of this set, such that no pair is contained in a block and, if two pairs cross, the four elements are in four distinct blocks.

Example 2.3 (Alternating tableaux). Let $\mathfrak{gl}_n$ be the adjoint representation of $\mathrm{GL}(n)$. This representation is not minuscule, but we can identify it with $V \otimes V^*$ where V is the vector representation of $\mathrm{GL}(n)$ and V^* is its dual. Let C_i, for $1 \leqslant i \leqslant r$, be the crystal corresponding to $V \otimes V^*$. Thus, the letters of a highest weight word w are pairs $(e_k, \text{-}e_\ell)$ of weight $e_k - e_\ell$, with $1 \leqslant k, \ell \leqslant n$. The corresponding $\mathrm{GL}(n)$-alternating tableau is the sequence of dominant weights $\varnothing = \mu^0, \mu^1, \dots, \mu^{2r} = \mu$, where $\mu^{2q} = \sum_{i=1}^{q} \mathrm{wt}(w_i)$, and $\mu^{2q+1} = \mu^{2q} + e_k$ when $w_{q+1} = (e_k, \text{-}e_\ell)$.

A word w is of highest weight if and only if μ^p is dominant for $p \leqslant 2r$. Given a $\mathrm{GL}(n)$-alternating tableau, one can recover the letter w_i of the corresponding highest weight word as $w_i = (e_k, \text{-}e_\ell)$ with $e_k = \mu^{2i-1} - \mu^{2i-2}$ and $\text{-}e_\ell = \mu^{2i} - \mu^{2i-1}$.

For example, the $\mathrm{GL}(2)$-alternating tableaux of length two are

$$(00, 10, 00, 10, 00), \quad (00, 10, 00, 10, 1\bar{1}), \quad (00, 10, 1\bar{1}, 10, 00),$$
$$(00, 10, 1\bar{1}, 10, 1\bar{1}), \quad (00, 10, 1\bar{1}, 2\bar{1}, 1\bar{1}), \quad (00, 10, 1\bar{1}, 2\bar{1}, 2\bar{2}),$$

writing $\bar{1}$ in place of -1, etc., for better readability. The corresponding highest weight words are

$$(e_1, \text{-}e_1)\,(e_1, \text{-}e_1), \quad (e_1, \text{-}e_1)\,(e_1, \text{-}e_2), \quad (e_1, \text{-}e_2)\,(e_2, \text{-}e_1),$$
$$(e_1, \text{-}e_2)\,(e_2, \text{-}e_2), \quad (e_1, \text{-}e_2)\,(e_1, \text{-}e_1), \quad (e_1, \text{-}e_2)\,(e_1, \text{-}e_2).$$

For n large enough, a suitable set of chord diagrams is the set of permutations of $\{1, \dots, r\}$, see Section 3.2 for definitions. We provide a bijection between this set and $\mathrm{GL}(n)$-alternating tableaux of empty shape, see Theorem 3.7 below. This bijection intertwines rotation with promotion, and reverse-complement with evacuation.

Example 2.4 (Fans of Dyck paths). Let S be the spin representation of the spin group $\mathrm{Spin}(2n+1)$ and let $\lambda_i = \frac{1}{2}\sum_{j=1}^{n} e_j$ be its dominant weight. Then w is a highest weight word if and only if $w_i = (\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ and μ^q is dominant for all $q \leqslant r$.

Therefore, a highest weight word w of weight zero can be identified with a fan of n Dyck paths of length r: the first entry of w_i is $\frac{1}{2}$ if and only if the top most Dyck path has an up-step at position i. In general, the j-th entry of w_i is $\frac{1}{2}$ if and only if the j-th Dyck path has an up-step at position i. One can show that ev acts on these as reversal.

In general, no suitable set of chord diagrams is known. For $n = 2$ there is an exceptional isomorphism between S and the vector representation of $\mathrm{Sp}(4)$. Thus, the results for oscillating tableaux apply in this case.

Example 2.5 (Vacillating tableaux). Let V be the vector representation of $\mathrm{SO}(2n+1)$ and let $\lambda_i = e_1$ be its dominant weight. Then w is a highest weight word if and only if w_i is in $\{\pm e_j : 1 \leqslant j \leqslant n\} \cup \{0\}$, μ^q is dominant for $q \leqslant r$ and $w_i \neq 0$ if μ^{i-1} contains an entry equal to 0. The corresponding tableaux are called vacillating tableaux and can be identified with n-fans of Riordan paths, see [8].

In general, no suitable set of chord diagrams is known. For $n = 1$ there is an exceptional isomorphism between V and the adjoint representation of $\mathrm{SL}(2)$. Thus we obtain a bijection between noncrossing set partitions without singletons and highest weight words of weight zero with the desired properties from the results in Section 3.2 below. For $n = 2$, a bijection between a basis of the invariant subspace of $\otimes^r V$ and certain chord diagrams follows from Kuperberg's webs [10].

3. Results

In this section we present combinatorial realisations of promotion and evacuation on highest weight words for the vector representation of the symplectic group and the adjoint representation of the general linear group. As it turns out, the former will essentially follow from the latter. The proofs are delegated to Section 6.

3.1. The vector representation of the symplectic groups.

Let us first recall Sundaram's definition of n-symplectic oscillating tableaux from example 2.2. As explained there, these are in bijection with the highest weight words in a tensor power of the crystal corresponding to the vector representation of the symplectic group $\mathrm{Sp}(2n)$.

Definition 3.1 (Sundaram [23]). *An n-symplectic oscillating tableau $\mathcal{O}$ of length r and (final) shape μ is a sequence of partitions*

$$\mathcal{O} = (\varnothing = \mu^0, \mu^1, \ldots, \mu^r = \mu)$$

such that the Ferrers diagrams of two consecutive partitions differ by exactly one cell, and each partition μ^i has at most n non-zero parts.

Recall that a *partial standard Young tableau* is a filling of the Ferrers diagram with distinct non-negative integers such that entries in rows and columns are strictly increasing.

A now classic bijection due to Sundaram [23] maps an oscillating tableau $\mathcal{O}$ of length r and shape μ to a pair $\big(\mathcal{M}(\mathcal{O}), \mathcal{M}_T(\mathcal{O})\big)$, consisting of a perfect matching of a subset of $\{1, \ldots, r\}$ and a partial standard Young tableau of shape μ, whose entries form the complementary subset. We present Roby's [16] description of this bijection in Section 5.

The *reversal* $\operatorname{rev} m$ of a perfect matching m of a subset of $\{1, \ldots, r\}$ is obtained by replacing each pair (i, j) with $(r + 1 - j, r + 1 - i)$. Our first main result relates this operation to evacuation as follows.

Theorem 3.2. *Let $\mathcal{O}$ be an n-symplectic oscillating tableau. Then $\mathcal{M}(\operatorname{ev}\mathcal{O})$ is the reversal of $\mathcal{M}(\mathcal{O})$ and $\mathcal{M}_T(\operatorname{ev}\mathcal{O})$ is the Schützenberger evacuation of $\mathcal{M}_T(\mathcal{O})$.*

If the oscillating tableau $\mathcal{O}$ is of empty shape the tableau $\mathcal{M}_T(\mathcal{O})$ is empty and $\mathcal{M}(\mathcal{O})$ is a perfect matching of $\{1, \ldots, r\}$. We use chord diagrams to visualise perfect matchings of $\{1, \ldots, r\}$, drawing their pairs as (straight) diagonals connecting the vertices of a counterclockwise labelled regular r-gon, see Figure 1.

A perfect matching is *(n+1)-noncrossing* if it contains at most n pairs that mutually cross in its chord diagram. It follows from Sundaram's bijection that these are precisely the perfect matchings corresponding to n-symplectic oscillating tableaux of empty shape.

Using the visualisation as a chord diagram, the *reversal* of a perfect matching is obtained by reflecting the diagonals of the diagram on a well-chosen axis. The *rotation* $\operatorname{rot} m$ of a matching m is obtained by replacing each pair (i, j) with the pair $\big(i \pmod r{+}1, j \pmod r{+}1\big)$. Visually, this corresponds to a rotation of the diagonals of the diagram.

Theorem 3.3. *The bijection $\mathcal{M}$ between n-symplectic oscillating tableaux of empty shape and $(n+1)$-noncrossing perfect matchings intertwines promotion and rotation, and evacuation and reversal:*

$$\operatorname{rot}\mathcal{M}(\mathcal{O}) = \mathcal{M}(\operatorname{pr}\mathcal{O}) \text{ and } \operatorname{rev}\mathcal{M}(\mathcal{O}) = \mathcal{M}(\operatorname{ev}\mathcal{O}).$$

Remark 3.4. Consider the natural embedding ι of the set of n-symplectic oscillating tableaux into the set of $(n+1)$-symplectic oscillating tableaux. Since rotation and reversal preserve the maximal cardinality of a crossing set of diagonals in the chord diagram, we obtain the remarkable fact that $\operatorname{pr}\iota(\mathcal{O}) = \iota(\operatorname{pr}\mathcal{O})$ and $\operatorname{ev}\iota(\mathcal{O}) = \iota(\operatorname{ev}\mathcal{O})$.

3.2. The adjoint representation of the general linear groups. We recall from example 2.3 Stembridge's definition of GL(n)-alternating tableaux, which are in bijection with the highest weight words in a tensor power of the crystal corresponding to the adjoint representation of GL(n).

Definition 3.5 (Stembridge [22]). *A staircase is a dominant weight of* GL(n)*, that is, a vector in $\mathbb{Z}^n$ with weakly decreasing entries. A* GL(n)-alternating tableau *$\mathcal{A}$ of length r and shape μ is a sequence of staircases*

$$\mathcal{A} = (\varnothing = \mu^0, \mu^1, \dots, \mu^{2r} = \mu)$$

such that

> *for even i, μ^{i+1} is obtained from μ^i by adding 1 to an entry, and*
> *for odd i, μ^{i+1} is obtained from μ^i by subtracting 1 from an entry.*

In Section 5, we introduce a bijection similar in spirit to Sundaram's. It maps an alternating tableau $\mathcal{A}$ of length r and shape μ to a triple $\big(\mathcal{P}(\mathcal{A}), \mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$, consisting of a bijection $\mathcal{P}(\mathcal{A}) : R \to S$ between two subsets of $\{1, \dots, r\}$, and two partial standard Young tableaux $\mathcal{P}_P(\mathcal{A})$ and $\mathcal{P}_Q(\mathcal{A})$. The shapes of these tableaux are obtained by separating the positive and negative entries of μ. The entries of the first tableau then form the complementary subset of R, the entries of the second form the complementary subset of S.

For a bijection $\pi : R \to S$ between two subsets of $\{1, \dots, r\}$, the *reverse-complement* $\operatorname{rc}\pi$ maps $r + 1 - i$ to $r + 1 - \pi(i)$.

Theorem 3.6. *Let $\mathcal{A}$ be a* GL(n)*-alternating tableau of length $r \leqslant \lfloor \frac{n+1}{2} \rfloor$. Then $\mathcal{P}(\operatorname{ev}\mathcal{A})$ is the reverse-complement of $\mathcal{P}(\mathcal{A})$, and $\mathcal{P}_P(\operatorname{ev}\mathcal{A})$ and $\mathcal{P}_Q(\operatorname{ev}\mathcal{A})$ are obtained by applying Schützenberger's evacuation to $\mathcal{P}_P(\mathcal{A})$ and $\mathcal{P}_Q(\mathcal{A})$ respectively:*

$$\big(\mathcal{P}_P(\operatorname{ev}\mathcal{A}), \mathcal{P}_Q(\operatorname{ev}\mathcal{A})\big) = \big(\operatorname{ev}\mathcal{P}_P(\mathcal{A}), \operatorname{ev}\mathcal{P}_Q(\mathcal{A})\big).$$

Similarly to Sundaram's map between oscillating tableaux and matchings, if the alternating tableau $\mathcal{A}$ is of empty shape the tableaux $\mathcal{P}_P(\mathcal{A})$ and $\mathcal{P}_Q(\mathcal{A})$ are empty and $\mathcal{P}(\mathcal{A})$ is a permutation. We use chord diagrams to visualise a permutation π, drawing an arc from vertex i to vertex $\pi(i)$ in a counterclockwise labelled regular r-gon, see Figure 1.

Using this visualisation, the *reverse-complement* of a permutation is obtained by reflecting the diagonals of the diagram on a well-chosen axis. The *rotation* $\operatorname{rot}\pi$ of a permutation π is obtained by replacing each arc $\big(i, \pi(i)\big)$ with the arc $\big(i \pmod r + 1, \pi(i) \pmod r + 1\big)$. Visually, this corresponds to a rotation of the diagonals of the diagram.

THEOREM 3.7. *For $n \geqslant r-1$ and also for $n \leqslant 2$ the bijection $\mathcal{P}$ between $\mathrm{GL}(n)$-alternating tableaux of empty shape of length r and permutations intertwines promotion and rotation:*

$$\operatorname{rot}\mathcal{P}(\mathcal{A}) = \mathcal{P}(\operatorname{pr}\mathcal{A}).$$

For even $n \geqslant r$ and for odd $n \geqslant r-1$, it intertwines evacuation and reverse-complement:

$$\operatorname{rc}\mathcal{P}(\mathcal{A}) = \mathcal{P}(\operatorname{ev}\mathcal{A}).$$

For $n \leqslant 2$ and arbitrary r, it intertwines evacuation and inverse-reverse-complement:

$$\operatorname{rc}\mathcal{P}(\mathcal{A})^{-1} = \mathcal{P}(\operatorname{ev}\mathcal{A}).$$

REMARK 3.8. Note that the case $n \leqslant 2$ is special. As we will show in Section 6.4, our bijection identifies $\mathrm{GL}(2)$-alternating tableaux of empty shape in a natural way with noncrossing partitions, which form an invariant set under rotation. In fact, this set coincides with the web basis for $\mathrm{GL}(2)$. Moreover, in this case the evacuation of an alternating tableau is simply its reversal. In terms of noncrossing set partitions, the inverse-reverse-complement of the corresponding permutation is the mirror image of the set partition.

REMARK 3.9. It is tempting to regard a $\mathrm{GL}(n)$-alternating tableau as a sequence of pairs of partitions by separating the positive and negative entries. Indeed, this is what we will do in Sections 5 and 6 to define our bijection and prove Theorems 3.6 and 3.7.

One might then think that promotion can be defined directly in terms of these sequences without reference to n. However, this is not the case. For $n > 2$, promotion does not preserve the maximal number of non-zero entries in a vector of an alternating tableau.

In fact, it is not clear whether there is an embedding ι of the set of $\mathrm{GL}(n)$-alternating tableaux in the set of $\mathrm{GL}(n+1)$-alternating tableaux such that $\operatorname{pr}\iota(\mathcal{A}) = \iota(\operatorname{pr}\mathcal{A})$. In spite of this, we prove a certain stability phenomenon for promotion of alternating tableaux in Theorem 6.1.

4. THE CACTUS GROUPS, LOCAL RULES, PROMOTION AND EVACUATION

In this section, following Henriques and Kamnitzer [7], we define promotion and evacuation of highest weight words as an action of certain elements of the r-fruit cactus group on r-fold tensor products of crystals. Then, following van Leeuwen [24] and Lenart [11] we encode the action of the cactus group by certain local rules, generalising Fomin's.

4.1. THE CACTUS GROUP AND ITS ACTION.

Let us first define the cactus groups.

DEFINITION 4.1. *The r-fruit cactus group, $\mathfrak{C}_r$, has generators $\mathrm{s}_{p,q}$ for $1 \leqslant p < q \leqslant r$ and defining relations*

- $\mathrm{s}_{p,q}^2 = 1$
- $\mathrm{s}_{p,q}\ \mathrm{s}_{k,l} = \mathrm{s}_{k,l}\ \mathrm{s}_{p,q}$ *if $q < k$ or $l < p$*
- $\mathrm{s}_{p,q}\ \mathrm{s}_{k,l} = \mathrm{s}_{p+q-l,p+q-k}\ \mathrm{s}_{p,q}$ *if $p \leqslant k < l \leqslant q$*

For convenience we additionally define $\mathrm{s}_{p,p} = 1$.

It may be useful to think of the generators as being indexed by intervals in $\{1,\dots,r\}$. Then the second relation can be rephrased by saying that two generators commute if they are indexed by disjoint intervals. The third relation is applicable when one interval is contained in the other, in which case the inner interval is reflected within the outer interval.

The following lemma shows that it is sufficient to define the action of the composites $\mathrm{s}_{1,q}\,\mathrm{s}_{2,q}$ for $2 \leqslant q \leqslant r$. The first relation was observed by White [26, Lem. 2.3], the second is in analogy to Schützenberger's original definition of evacuation of standard Young tableaux in [20, Sec. 5].

Lemma 4.2. *We have*

$$\mathrm{s}_{p,q} = \mathrm{s}_{1,q}\,\mathrm{s}_{1,q-p+1}\,\mathrm{s}_{1,q} \quad \text{and} \quad \mathrm{s}_{1,q} = \mathrm{s}_{1,2}\,\mathrm{s}_{2,2}\ \mathrm{s}_{1,3}\,\mathrm{s}_{2,3}\ \dots\ \mathrm{s}_{1,q}\,\mathrm{s}_{2,q}\,.$$

Proof. The first equality is obtained from the third defining relation by replacing p,q,k,ℓ with $1,q,p$ and q respectively. The second equality follows from $\mathrm{s}_{1,\ell}\,\mathrm{s}_{2,\ell} = \mathrm{s}_{1,\ell-1}\,\mathrm{s}_{1,\ell}$, which is also an instance of the third defining relation. □

Henriques and Kamnitzer [7] defined an action of the cactus group on r-fold tensor products of crystals in terms of the commutor, which in turn is defined using Lusztig's involution. Let us first briefly recall the latter, as introduced in [12].

Definition 4.3. *Let B be a crystal graph associated with an irreducible representation. Lusztig's involution η maps the unique highest weight vertex of B to its unique lowest weight vertex, and the Kashiwara lowering operator f_i to the Kashiwara raising operator e_{i^*}, where $i \mapsto i^*$ is the Dynkin diagram automorphism specified by $\alpha_{i^*} = -w_0(\alpha_i)$, and w_0 is the longest element of the Weyl group. This definition is extended to arbitrary crystals by applying the involution to each connected component separately.*

Lusztig's involution is not a morphism of crystals, which would have to map highest weight vertices to highest weight vertices. For the Cartan type A_n crystal B_λ of semistandard Young tableaux of shape λ, Lusztig's involution is precisely Schützenberger's evacuation of semistandard Young tableaux with largest entry at most $n+1$.

Definition 4.4. *For two crystals A and B, the commutor is the crystal morphism*

$$\begin{aligned} \sigma_{A,B} &: A \otimes B \to B \otimes A \\ a\,b &\mapsto \eta\big(\eta(b)\,\eta(a)\big). \end{aligned}$$

We can now define the action of the cactus group.

Definition 4.5. *The action of $\mathfrak{C}_r$ on words in $C_1 \otimes \dots \otimes C_r$ is defined inductively by letting $\mathrm{s}_{p,p+1}$ act as $1 \otimes \sigma_{C_p,C_{p+1}} \otimes 1$ and $\mathrm{s}_{p,q}$ as $(1 \otimes \sigma_{C_p,C_{p+1}\otimes\dots\otimes C_q} \otimes 1) \circ \mathrm{s}_{p+1,q}$ for $q > p+1$, where 1 denotes the identity map on a crystal.*

The action can be expressed more explicitly directly in terms of Lusztig's involution.

Proposition 4.6. *Let $w = w_1 \dots w_r$ be a word in $C_1 \otimes \dots \otimes C_r$, then*

$$\mathrm{s}_{p,q}\,w_1 \dots w_r = w_1 \dots w_{p-1}\eta\big(\eta(w_q)\eta(w_{q-1}) \dots \eta(w_p)\big)w_{q+1} \dots w_r.$$

Proof. This follows by induction on $q-p$ and the fact that η is an involution. □

Definition 4.7. *The promotion $\mathrm{pr}\,w$ of w is $\mathrm{s}_{1,r}\,\mathrm{s}_{2,r}\,w$, and the evacuation $\mathrm{ev}\,w$ of w is $\mathrm{s}_{1,r}\,w$. For a tableau $\mathcal{T}$ corresponding to a highest weight word w, we use $\mathrm{pr}\,\mathcal{T}$ and $\mathrm{ev}\,\mathcal{T}$ to denote the tableaux corresponding to $\mathrm{pr}\,w$ and $\mathrm{ev}\,w$.*

It will be convenient to express promotion as a commutor, as follows.

PROPOSITION 4.8. $\mathrm{s}_{1,q}\,\mathrm{s}_{2,q}(w) = \sigma_{C_1,C_2\otimes\cdots\otimes C_q}(w)$.

Proof. This is immediate from the definition of the action of $\mathrm{s}_{1,q}$ and from the fact that $\mathrm{s}_{2,q}^2 = 1$. □

4.2. PROMOTION AND EVACUATION VIA LOCAL RULES. We now follow Lenart's approach [11] and realise the action of the cactus group using van Leeuwen's local rules [24, Rule 4.1.1], which generalise Fomin's [21, A 1.2.7].

From now on we restrict ourselves to minuscule representations. However, let us remark that van Leeuwen also provides a local rule that applies to quasi-minuscule representations, which makes our approach viable for arbitrary representations.

For minuscule representations, van Leeuwen's rules involve obtaining the unique dominant representative of a weight.

DEFINITION 4.9. *Let λ be a weight of a representation of a Lie group with Weyl group W. Then $\mathrm{dom}_W(\lambda)$ is the unique dominant representative of the W-orbit $W\lambda$.*

EXAMPLE 4.10. The Weyl group of $\mathrm{SL}(n)$ is the symmetric group $\mathfrak{S}_n$. Thus, $\mathrm{dom}_{\mathfrak{S}_n}$ returns its argument sorted into decreasing order.

EXAMPLE 4.11. The Weyl group of $\mathrm{Sp}(2n)$ is the hyperoctahedral group $\mathfrak{H}_n$ of signed permutations of $\{\pm1,\dots,\pm n\}$. Thus, the dominant representative $\mathrm{dom}_{\mathfrak{H}_n}(\lambda)$ of a weight λ is obtained by sorting the absolute values of its entries into decreasing order.

We can now define the local rule.

DEFINITION 4.12. *Let A be a crystal and B and C be crystals of minuscule representations. Then the local rule*

$$\tau^A_{B,C}\colon A\otimes B\otimes C \to A\otimes C\otimes B$$

is an isomorphism of crystals defined for highest weight words $a\,b\,c$ as follows: let κ be the weight of a, let λ be the weight of $a\,b$ and let ν be the weight of $a\,b\,c$. Then

$$\tau^A_{B,C}(a\,b\,c) = a\,\hat{c}\,\hat{b},$$

where

$$\hat{c} = \mu - \kappa \quad \textit{and} \quad \hat{b} = \nu - \mu \quad \textit{with} \quad \mu = \mathrm{dom}_W(\kappa+\nu-\lambda).$$

We represent this by the following diagram:

(1)
$$\begin{array}{ccc} \lambda & \xrightarrow{c} & \nu \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle \hat{b}} \\ \kappa & \xrightarrow[\hat{c}]{} & \mu \end{array}$$

Since any isomorphism between crystals is determined by specifying a bijection between the corresponding highest weight words, this definition can be extended to an isomorphism between $A\otimes B\otimes C$ and $A\otimes C\otimes B$ by applying the lowering operators.

REMARK 4.13. When $B = C$ is the crystal associated with the vector representation of $\mathrm{SL}(n)$, the local rule (1) is Fomin's for Schützenberger's jeu de taquin [21, A 1.2.7]. More explicitly, if λ is the only partition of its size that contains κ and is contained in ν, then $\mu = \lambda$. Otherwise there is a unique such partition different from λ, and this is μ.

REMARK 4.14. As in the classical case the local rule is symmetric in the sense that $\mu = \mathrm{dom}_W(\kappa+\nu-\lambda)$ if and only if $\lambda = \mathrm{dom}_W(\kappa+\nu-\mu)$, see [24, Lem. 4.1.2].

Example 4.15. Let C be the crystal associated with the vector representation of SL(3), and let A and B be the crystal associated with its exterior square. Let $a = e_1+e_2$, let $b = e_1+e_3$ and let $c = e_2$. Then $a\,b\,c$ is a highest weight word in $A\otimes B\otimes C$. We have $\kappa = (1,1,0)$, $\lambda = (2,1,1)$ and $\nu = (2,2,1)$.

Thus, since $\mathrm{dom}_{\mathfrak{S}_n}$ sorts its argument into decreasing order,

$$\mu = \mathrm{dom}_{\mathfrak{S}_n}(\kappa+\nu-\lambda) = \mathrm{dom}_{\mathfrak{S}_n}(1,2,0) = (2,1,0), \quad \text{and} \quad \hat{b} = e_2+e_3, \quad \hat{c} = e_1.$$

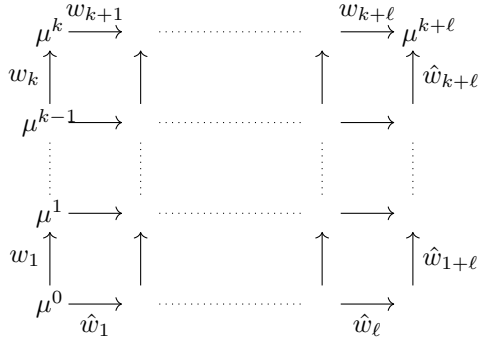
Example 4.16. Let B and C be the crystal associated with the vector representation of Sp(4) and let $A = \otimes^2 B$. Let $a = e_1\,e_1$, let $b = e_2$ and let $c = -e_2$. Then $a\,b\,c$ is a highest weight word in $A\otimes B\otimes C$. We have $\kappa = (2,0)$, $\lambda = (2,1)$ and $\nu = (2,0)$.

Thus, since $\mathrm{dom}_{\mathfrak{H}_n}$ takes absolute values and then sorts into decreasing order,

$$\mu = \mathrm{dom}_{\mathfrak{H}_n}(\kappa+\nu-\lambda) = \mathrm{dom}_{\mathfrak{H}_n}(2,-1) = (2,1), \quad \text{and} \quad \hat{b} = b, \quad \hat{c} = c.$$

Theorem 4.17 ([11, Thm. 4.4]). *Let A and B be crystals embedded into tensor products $A_1\otimes\cdots\otimes A_k$ and $B_1\otimes\cdots\otimes B_\ell$ of crystals of minuscule representations. Let $w = w_1,\dots,w_{k+\ell}$ be a highest weight word in $A\otimes B$ with corresponding tableau $\varnothing = \mu^0,\mu^1,\dots,\mu^r = \mu$. Then $\sigma_{A,B}(w)$ can be computed as follows. Create a $k\times\ell$ grid of squares as in* (1)*, labelling the edges along the left border with $w_1,\dots,w_k$ and the edges along the top border with $w_{k+1},\dots,w_{k+\ell}$:*

(2)



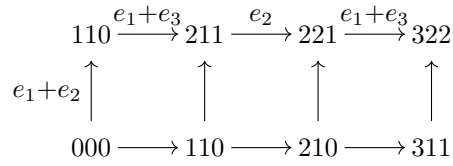
For each square whose left and top edges are already labelled use the local rule to compute the labels on the square's bottom and right edges. The labels $\hat{w}_1\dots\hat{w}_{k+\ell}$ of the edges along the bottom and the right border of the grid then form $\sigma_{A,B}(w)$.

Example 4.18. Let $C_1 = C_2 = C_4$ be the crystal associated with the exterior square of SL(3) and let C_3 the crystal corresponding to its vector representation. Then $w = e_1{+}e_2\ e_1{+}e_3\ e_2\ e_1{+}e_3$ is the highest weight word corresponding to the semistandard tableau

1	1	2
2	3	4
4		

.

The promotion $\mathrm{pr}\,w = \sigma_{C_1,C_2\otimes C_3\otimes C_4}(w)$ can now be computed using Theorem 4.17. We put the zero weight in the bottom-left corner. Then, beginning with $\mu^1 = w_1$, we write down the sequence of cumulative weights $\mu^1,\dots,\mu^4$, with $\mu^q = \sum_{i=1}^q w_i$ in the top row. Finally, we successively apply the local rule (1) and fill in the weights in the bottom row.



From now on we omit edges and their labels, because the labels are determined by the weights.

By Theorem 4.17 the promotion of w is given by the sequence of weights in the bottom row, together with the weight in the top-right corner. The corresponding tableau is

1	1	3
2	4	4
3		

.

As mentioned in the introduction, this definition of promotion coincides with the classical definition of promotion in terms of Bender–Knuth moves on tableaux when the crystals correspond to exterior powers of the vector representation of $\mathrm{SL}(n)$.

As an aside we obtain a formulation of the commutor, and therefore also of promotion of highest weight words in crystals of minuscule representations, analogous to the definition in terms of slides in tableaux, as follows.

Corollary 4.19. *Let A and B be crystals embedded into tensor products of crystals of minuscule representations. Let $a \in A$ and $b \in B$ such that $a\,b$ is a highest weight word in $A \otimes B$, and let $\hat{b}\,\hat{a} = \sigma_{A,B}(a\,b)$ with $\hat{b} \in B$ and $\hat{a} \in A$.*

Then

$\hat{b}$ is the highest weight vertex in the same component of B as b, and
$\hat{a}$ is a vertex of A such that the weight of $\hat{b}\,\hat{a}$ equals the weight of $a\,b$.

In particular, if A is a crystal of a minuscule representation, $\hat{a}$ is determined uniquely by its weight.

Proof. Let B_λ be the component of B containing b. Because of the naturality of the commutor in B (see property $(C1)$ in [11]), $\sigma_{A,B}(a\,b)$ equals $\sigma_{A,B_\lambda}(a\,b)$.

Since $a\,b$ is a highest weight word and the commutor is an isomorphism of crystals, $\sigma_{A,B_\lambda}(a\,b) = \hat{b}\,\hat{a}$ is also a highest weight word. It follows that $\hat{b}$ is of highest weight, and therefore equals the highest weight vertex of B_λ. □

4.3. Promotion and evacuation of $\mathrm{GL}(n)$-alternating tableaux. Let us now make promotion and evacuation of $\mathrm{GL}(n)$-alternating tableaux explicit. Recall from example 2.3 that we regard the adjoint representation as the tensor product $V \otimes V^*$, where V is the vector representation of $\mathrm{GL}(n)$ and V^* is its dual. Both of these are minuscule, so we can apply Theorem 4.17.

Therefore, the rectangular grid to compute $\mathrm{pr}\,w = \sigma_{C_1,C_2\otimes\cdots\otimes C_r}(w)$ has three rows, the bottom row begins with the zero weight, the middle row with μ^1, which is always e_1, and the top row contains the remaining cumulative weights $\mu^2, \ldots, \mu^{2r}$.

However, because we will apply promotion repeatedly, it will be convenient to slightly enlarge this grid, and prepend the two weights μ^0 and μ^1 to the first row. After having successively applied the local rule (1) and thus computed the remaining weights in the middle and the bottom row, we append the final element of the second row and the weight of the original word to the third line. We call the resulting diagram the *promotion diagram* of an alternating tableau:

$$
\begin{array}{llllll}
\mu^{0=\varnothing} & \mu^{1=1} & \mu^2 & \ldots\ldots & \mu^{2r} & \\
 & & \acute{\mu}^{1=\mu^1} & \ldots\ldots & \acute{\mu}^{2r-1} & \\
 & & \hat{\mu}^{0=\mu^0} & \ldots\ldots & \hat{\mu}^{2r-2} & \hat{\mu}^{2r-1=\acute{\mu}^{2r-1}}\ \hat{\mu}^{2r=\mu^{2r}}
\end{array}
\tag{3}
$$

To illustrate, let us compute the promotion of the $\mathrm{GL}(3)$ highest weight word $w = (e_1, \text{-}e_3)\,(e_1, \text{-}e_2)\,(e_2, \text{-}e_2)\,(e_2, \text{-}e_1)\,(e_3, \text{-}e_1)$. The first row is the alternating tableau

corresponding to w. For better readability we write $\bar{1}$ in place of -1.

$$\begin{array}{r} 000\ 100\ 10\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1}\ 20\bar{1}\ 10\bar{1}\ 100\ 000 \\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 20\bar{1}\ 21\bar{1}\ 11\bar{1}\ 110\ 100 \\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}\ 100\ 10\bar{1}\ 100\ 000 \end{array} \tag{4}$$

Thus, the promotion of w is $\operatorname{pr} w = (e_1, \text{-}e_3)\,(e_2, \text{-}e_2)\,(e_2, \text{-}e_2)\,(e_3, \text{-}e_3)\,(e_3, \text{-}e_1)$.

The six vectors in the rectangle in diagram (4) demonstrate that the naive embedding of $\mathrm{GL}(n)$-alternating tableaux into the set of $\mathrm{GL}(n{+}1)$-alternating tableaux is not compatible with promotion, as already mentioned in Section 3.2: padding the vectors of the original word with zeros, and applying the local rule, we obtain the rectangle

$$\begin{array}{c} 200\bar{1}\ 20\bar{1}\bar{1} \\ 210\bar{1}\ 21\bar{1}\bar{1} \\ 110\bar{1}\ 11\bar{1}\bar{1}, \end{array}$$

with bottom-right vector $11\bar{1}\bar{1}$, rather than $100\bar{1}$ as one might expect.

$$\begin{array}{rl}
\mathcal{A} = & 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 2\bar{1}\bar{1}\ 3\bar{1}\bar{1}\ 2\bar{1}\bar{1}\ 20\bar{1}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 31\bar{3}\ 21\bar{3} \\
& 100\ 200\ 20\bar{1}\ 200\ 20\bar{1}\ 30\bar{1}\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 31\bar{2}\ 31\bar{3}\ 32\bar{3}\ 22\bar{3} \\
& 000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 21\bar{2}\ 21\bar{3}\ 22\bar{3}\ 21\bar{3} \\
& 100\ 110\ 11\bar{1}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 31\bar{2}\ 31\bar{3}\ 32\bar{3}\ 31\bar{3} \\
& 000\ 100\ 10\bar{1}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 31\bar{3}\ 30\bar{3} \\
& 100\ 200\ 100\ 200\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 31\bar{2}\ 30\bar{2} \\
& 000\ 100\ 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 20\bar{2} \\
& 100\ 200\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 31\bar{2}\ 30\bar{2} \\
& 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 20\bar{2} \\
& 100\ 200\ 20\bar{1}\ 21\bar{1}\ 20\bar{1} \\
& 000\ 100\ 10\bar{1}\ 11\bar{1}\ 10\bar{1} \\
& 100\ 110\ 100 \\
& 000\ 100\ 10\bar{1} \\
& 100 \\
& 000 \\
= & \operatorname{ev} \mathcal{A}
\end{array}$$

Figure 2. The evacuation of an alternating tableau.

To obtain the evacuation of an alternating tableau we use the second identity of Lemma 4.2. We start by computing the promotion of the initial alternating tableau as above, except that we do not append anything to the third row. We then repeat this process a total of r times, creating a (roughly) triangular array of weights, which we call the *evacuation diagram* of an alternating tableau. The sequence of cumulative weights of the evacuation can then be read off the vertical row on the right hand side, from bottom to top. An example can be found in Figure 2. The symbols $\oplus$, $\ominus$ and $\otimes$ occurring in the figure should be ignored for the moment.

Finally, we would like to point out that for alternating tableaux of empty shape there is a second way to compute the promotion, exploiting the fact that the next-to-last weight is forced to be $10\dots0$. Let $w = w_1 \dots w_r$ be the highest weight word corresponding to the alternating tableau. We consider w as an element of $A \otimes A^* \otimes B_\lambda$, where A is the crystal corresponding to V, A^* is the crystal corresponding to V^* and, similar to what was done in the proof of Corollary 4.19, B_λ is the component of $\bigotimes^{r-1}(A \otimes A^*)$ containing $w_3 \dots w_r$. Then we first compute $\hat{w} = \sigma_{A,A^*\otimes B_\lambda}(w)$, followed by computing $\hat{\hat{w}} = \sigma_{A^*,B_\lambda\otimes A}(\hat{w})$:

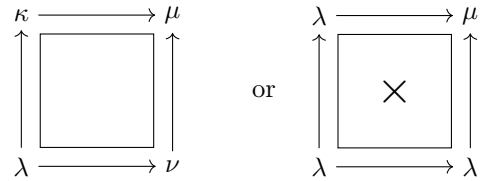
$$\begin{array}{c} 000\ 100\ 10\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1}\ 20\bar{1}\ 10\bar{1}\ 100\ 000 \\ 000\ 00\bar{1}\ 10\bar{1}\ 1\bar{1}\bar{1}\ 10\bar{1}\ 1\bar{1}\bar{1}\ 10\bar{1}\ 00\bar{1}\ 000\ 00\bar{1}\ 000 \\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}\ 100\ 10\bar{1}\ 100\ 000 \end{array} \tag{5}$$

Because the initial segment of $\hat{\hat{w}}$ is an element of B_λ and is of highest weight, it must coincide with the initial segment of the promotion of w.

This variant of the local rules for promotion was recently rediscovered, in slightly different form, by Patrias [13]. Note, however, that for an alternating tableau of non-empty shape, this procedure yields a tableau which, in general, is different from the result of promotion.

5. Growth diagram bijections

In this section we recall Sundaram's bijection (using Roby's description [16] based on Fomin's growth diagrams [6]) between oscillating tableaux and matchings. We also present a new bijection, in the same spirit, between alternating tableaux and partial permutations. In both cases, the action of the cactus group on highest weight words becomes particularly transparent when using Fomin's growth diagrams and local rules for the Robinson–Schensted correspondence.



forward rules: $\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$ $\qquad$ $\mu = \lambda + e_1$

backward rules: $\lambda' = \mathrm{sort}(\kappa' + \nu' - \mu')$ $\qquad$ $\lambda = \mu - e_1$

Figure 3. Cells of a growth diagram and corresponding local rules.

For our purposes, a growth diagram is a finite collection of cells, arranged in the form of a Ferrers diagram using the French convention, as for example in Figures 4

and 5. Let us first describe the classical setup, which we use to describe Sundaram's correspondence.

In this case, each cell is either empty or contains a cross. Moreover, we require that in every row and every column of the growth diagram there is at most one cell which contains a cross. Every corner of a cell is labelled with a partition such that the local rules in Figure 3 are satisfied, where λ' denotes the partition conjugate to λ and e_1 is the first unit vector. Moreover, we require that two adjacent partitions (as for example λ and κ in Figure 3) either coincide or the one at the head of the arrow is obtained from the other by adding a unit vector.

Furthermore, we require that the partitions labelling the corners of a cell satisfy the forward and backward rules of Figure 3. In fact, the two *forward rules* determine μ given the other three partitions and the content of the cell. The two *backward rules* determine the content of the cell and the partition λ in the bottom-left given the other three partition.

Thus, the information in a growth diagram is redundant. In particular, given the partitions labelling the corners along the bottom-left border and the contents of the cells, one can recover the remaining partitions. Conversely, given the partitions labelling the corners along the top-right border of a diagram, one can recover the remaining partitions and the contents of the cells.

The presentation of the local rules in Figure 3 is slightly non-standard. It has the benefit that the local rule for empty cells is very similar to the special case of Definition 4.12 corresponding to Cartan type A_n, with two important differences. The first difference is that all partitions are transposed, the second, that the orientation of the vertical arrows is reversed.

5.1. Roby's description of Sundaram's correspondence. In this section we recall the bijection between oscillating tableaux of length r and shape μ and pairs consisting of a partial matching of $\{1,\dots,r\}$ and a partial standard Young tableau of shape μ whose entries are the unmatched elements.

Definition 5.1. *Let $\mathcal{O} = (\mu_0, \mu_1, \dots, \mu_r)$ be an oscillating tableau. The associated (triangular) growth diagram $\mathcal{G}(\mathcal{O})$ consists of r left-justified rows, with $i-1$ cells in row i for $i \in \{1,\dots,r\}$, where row 1 is the top row. Label the cells according to the following specification:*

- R1 *Label the north east corners of the cells on the main diagonal from the top-left to the bottom-right with the partitions in $\mathcal{O}$.*
- R2 *Label the corners of the first subdiagonal with the smaller of the two partitions labelling the two adjacent corners on the diagonal.*
- R3 *Use the backward rules to determine which cells contain a cross.*

Let $\mathcal{M}(\mathcal{O})$ be the matching containing a pair $\{i,j\}$ for every cross in column i and row j of the $\mathcal{G}(\mathcal{O})$. Furthermore, let $\mathcal{M}_T(\mathcal{O})$ be the partial standard Young tableau corresponding to the sequence of partitions along the bottom border of $\mathcal{G}(\mathcal{O})$.

Theorem 5.2 (Sundaram [23, Sec. 8], Roby [16, Prop. 4.3.1]). *The map $\mathcal{O} \mapsto \big(\mathcal{M}(\mathcal{O}), \mathcal{M}_T(\mathcal{O})\big)$ is a bijection between oscillating tableaux of length r and shape μ, and pairs consisting of a perfect matching of a subset of $\{1,\dots,r\}$ and a partial standard Young tableau of shape μ, whose entries form the complementary subset.*

Moreover, the map $\mathcal{O} \mapsto \mathcal{M}(\mathcal{O})$ is a bijection between n-symplectic oscillating tableaux of length r and empty shape and $(n+1)$-noncrossing perfect matchings of $\{1,\dots,r\}$.

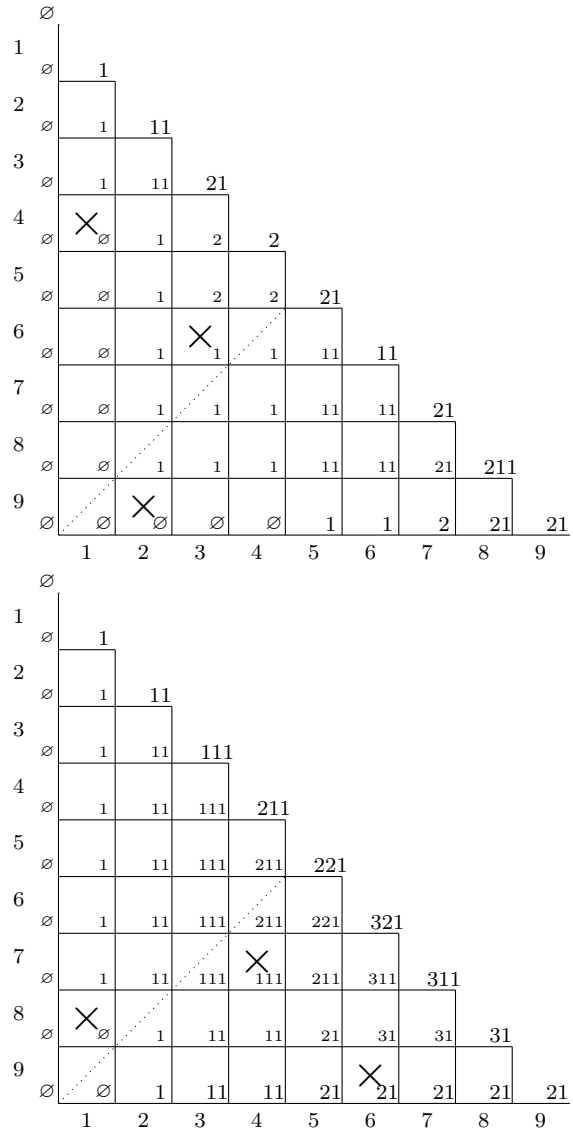


FIGURE 4. A pair of growth diagrams $\mathcal{G}(\mathcal{O})$ and $\mathcal{G}(\mathrm{s}_{1,9}\,\mathcal{O})$, with $\mathcal{O} = (\varnothing, 1, 11, 21, 2, 21, 11, 21, 211, 21)$, illustrating Theorem 3.2. The dotted line indicates the axis of reflection for the matchings $\mathcal{M}(\mathcal{O})$ and $\mathcal{M}(\mathrm{s}_{1,9}\,\mathcal{O})$.

An example for this procedure, which also illustrates Theorem 3.2, can be found in Figure 4. Let $\mathcal{O}$ be the 3-symplectic oscillating tableau

$$(\varnothing, 1, 11, 21, 2, 21, 11, 21, 211, 21),$$

whose partitions label the corners of the diagonal of the first growth diagram. Applying the backward rules, we obtain the matching and the partial standard Young tableau

$$\mathcal{M}(\mathcal{O}) = \big\{\{1,4\},\{2,9\},\{3,6\}\big\} \text{ and } \mathcal{M}_T(\mathcal{O}) = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 \\ \cline{1-1}\end{array}.$$

Using Lemma 4.2 and the local rule in Definition 4.12 one can compute that $\operatorname{ev}\mathcal{O}$ is the 3-symplectic oscillating tableau labelling the corners of the diagonal of the second growth diagram. Applying the backward rules again, we obtain the matching and the partial standard Young tableau predicted by Theorem 3.2:

$$\mathcal{M}(\operatorname{ev}\mathcal{O}) = \big\{\{1,8\},\{4,7\},\{6,9\}\big\} \text{ and } \mathcal{M}_T(\operatorname{ev}\mathcal{O}) = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

5.2. A new variant for Stembridge's alternating tableaux. In this section we present a variation of Sundaram's bijection for alternating tableaux and permutations.

Recall that a staircase is a vector with weakly decreasing integer entries. The *positive part* of the staircase is the partition obtained by removing all entries less than or equal to zero. The *negative part* of the staircase is the partition obtained by removing all entries greater than or equal to zero, removing the signs of the remaining entries and reversing the sequence.

Definition 5.3. *Let $\mathcal{A} = (\mu^0, \mu^1, \dots, \mu^{2r})$ be an alternating tableau. The associated growth diagram $\mathcal{G}(\mathcal{A})$ is an $r \times r$ square of cells, obtained as follows:*

- P1 *Label the north east corners of the cells on the main diagonal and the first superdiagonal from the top-left to the bottom-right with the staircases in $\mathcal{A}$.*
- P2 *Apply the backward rules on the positive parts of the staircases to determine which cells below the diagonal contain a cross.*
- P3 *Use the backward rules (rotated by 180°) on the negative parts of the staircases to determine which cells above the diagonal contain a cross.*

Let $\mathcal{P}(\mathcal{A})$ be the partial permutation mapping i to j for every cross in column i and row j of $\mathcal{G}(\mathcal{A})$, and let $\big(\mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$ be the pair of partial standard Young tableaux corresponding to the sequence of partitions along the bottom and the right border of $\mathcal{G}(\mathcal{A})$, respectively.

Theorem 5.4. *The map $\mathcal{A} \mapsto \big(\mathcal{P}(\mathcal{A}), \mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$ is a bijection between alternating tableaux of length r and shape μ, and triples consisting of a bijection $\mathcal{P}(\mathcal{A}) : R \to S$ between two subsets of $\{1, \dots, r\}$, and two partial standard Young tableaux $\mathcal{P}_P(\mathcal{A})$ and $\mathcal{P}_Q(\mathcal{A})$. The shapes of these tableaux are obtained by separating the positive and negative entries of μ. The entries of the first tableau then form the complementary subset of R, the entries of the second form the complementary subset of S.*

Moreover, the map $\mathcal{A} \mapsto \mathcal{P}(\mathcal{A})$ is a bijection between $\mathrm{GL}(n)$-alternating tableaux of length r and empty shape and permutations of $\{1, \dots, r\}$ whose longest increasing subsequence has length at most n.

An example for this procedure, which also illustrates Theorem 3.6, can be found in Figure 5. We render fixed points as $\otimes$, other crosses below the diagonal as $\oplus$ and crosses above the diagonal as $\ominus$. The reason for doing so is given by Corollary 6.19 in Section 6.3, where we show that the growth diagram of an alternating tableau and its evacuation diagram are very closely related.

Let $\mathcal{A}$ be the $\mathrm{GL}(13)$-alternating tableau of length 7

$$(\varnothing, 1, 1\bar{1}, 2\bar{1}, 2\bar{2}, 2\bar{1}, 2\bar{1}\bar{1}, 3\bar{1}\bar{1}, 2\bar{1}\bar{1}, 2\bar{1}, 2\bar{2}, 3\bar{2}, 3\bar{3}, 3\bar{1}\bar{3}, 2\bar{1}\bar{3}),$$

where we write the negative entries with bars and omit zeros for readability. Its staircases label the corners of the diagonal of the first growth diagram. Applying the backward rules we obtain the partial permutation and the partial standard Young tableaux

$$\mathcal{P}(\mathcal{A}) = \big\{(3,2),(4,4),(5,1),(6,7)\big\}, \mathcal{P}_P(\mathcal{A}) = \begin{array}{|c|c|c|}\hline 3 & 5 & 6 \\ \hline \end{array}, \text{ and } \mathcal{P}_Q(\mathcal{A}) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 7 \\ \cline{1-1}\end{array}.$$

The second growth diagram in the figure is obtained by applying the same procedure to $\operatorname{ev}\mathcal{A} = \mathrm{s}_{1,7}\,\mathcal{A}$, which yields

$$\mathcal{P}(\operatorname{ev}\mathcal{A}) = \big\{(2,1),(3,7),(4,4),(5,6)\big\}, \mathcal{P}_P(\operatorname{ev}\mathcal{A}) = \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline \end{array}, \text{ and } \mathcal{P}_Q(\operatorname{ev}\mathcal{A}) = \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}.$$

as predicted by Theorem 3.6.

In the example above, we could have obtained the same sequence of positive and negative parts of the staircases from a GL(3)-alternating tableau, removing ten zeros from each vector. As it turns out, evacuation of this alternating tableau yields the same result as above, although for $r = 7$ Theorem 3.6 applies only when n is at least 13. The computation of the evacuated alternating tableau is carried out in Figure 2.

The GL(2)-alternating tableau $\mathcal{A} = (\varnothing, 10, 1\bar{1}, 10, 1\bar{1})$ of length 2 illustrates the necessity of the hypothesis restricting the length of the alternating tableau in Theorem 3.6. On the one hand, this tableau is fixed by $\operatorname{ev} = \mathrm{s}_{1,2}$. On the other hand, $\mathcal{P}(\mathcal{A}) = \{(2,1)\}$, whose reverse-complement is $\{(1,2)\}$.

Similarly, to justify the necessity of the hypothesis in Theorem 3.7, consider the GL(3)-alternating tableau in the first row of diagram (4), which corresponds to the permutation depicted in Figure 1. Its promotion, as computed in the last row of diagram (4), corresponds to the permutation 23514, which differs from the rotated permutation.

6. Proofs

Our strategy is as follows. We first consider only GL(n)-alternating tableaux of empty shape and length r with $n \geqslant r$, and show that the bijection $\mathcal{P}$ presented in Section 5.2 intertwines rotation and promotion. To do so, we demonstrate that the middle row of the promotion diagram (3) of an alternating tableau $\mathcal{A}$ can be interpreted as corresponding to a single-step rotation of the rows of the growth diagram $\mathcal{G}(\mathcal{A})$. Then, using a very similar argument, we find that the promotion of $\mathcal{A}$ corresponds to a single-step rotation of the columns of the growth diagram just obtained.

To prove the statements concerning evacuation, we show that the permutation $\mathcal{P}(\mathcal{A})$ can actually be read off directly from the evacuation diagram. In particular, this makes the effect of evacuation on $\mathcal{P}(\mathcal{A})$ completely transparent. The effect of the evacuation of an arbitrary alternating tableau $\mathcal{A}$ on the triple $\big(\mathcal{P}(\mathcal{A}), \mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$ is deduced from the special case of alternating tableaux of empty shape by extending $\mathcal{A}$ to an alternating tableau of empty shape.

In order to determine the exact range of validity of Theorem 3.7 we use a stability phenomenon proved in Section 6.1. The case $n = 2$ is treated completely separately in Section 6.4.

Finally, in Section 6.5, we deduce the statements for oscillating tableaux and the vector representation of the symplectic groups, Theorem 3.2 and 3.3, from the statements for alternating tableaux.

6.1. Stability.

In this section we prove a stability phenomenon needed for establishing the exact bounds in Theorem 3.7. Given the lack of an embedding of GL(n)-alternating tableaux in the set of GL($n+1$)-alternating tableaux that is compatible with promotion, this theorem may be interesting in its own right.

Theorem 6.1. *Let $\mathcal{A}$ be a* GL(n)*-alternating tableau, not necessarily of empty shape, and suppose that each staircase in $\mathcal{A}$ and* $\operatorname{pr}\mathcal{A}$ *contains at most m nonzero parts.*

Then $\operatorname{pr}\tilde{\mathcal{A}} = \widetilde{\operatorname{pr}\mathcal{A}}$*, where $\tilde{\mathcal{A}}$ and $\widetilde{\operatorname{pr}\mathcal{A}}$ are the* GL(m)*-alternating tableaux obtained from $\mathcal{A}$ and* $\operatorname{pr}\mathcal{A}$ *by removing $n-m$ zeros from each staircase.*

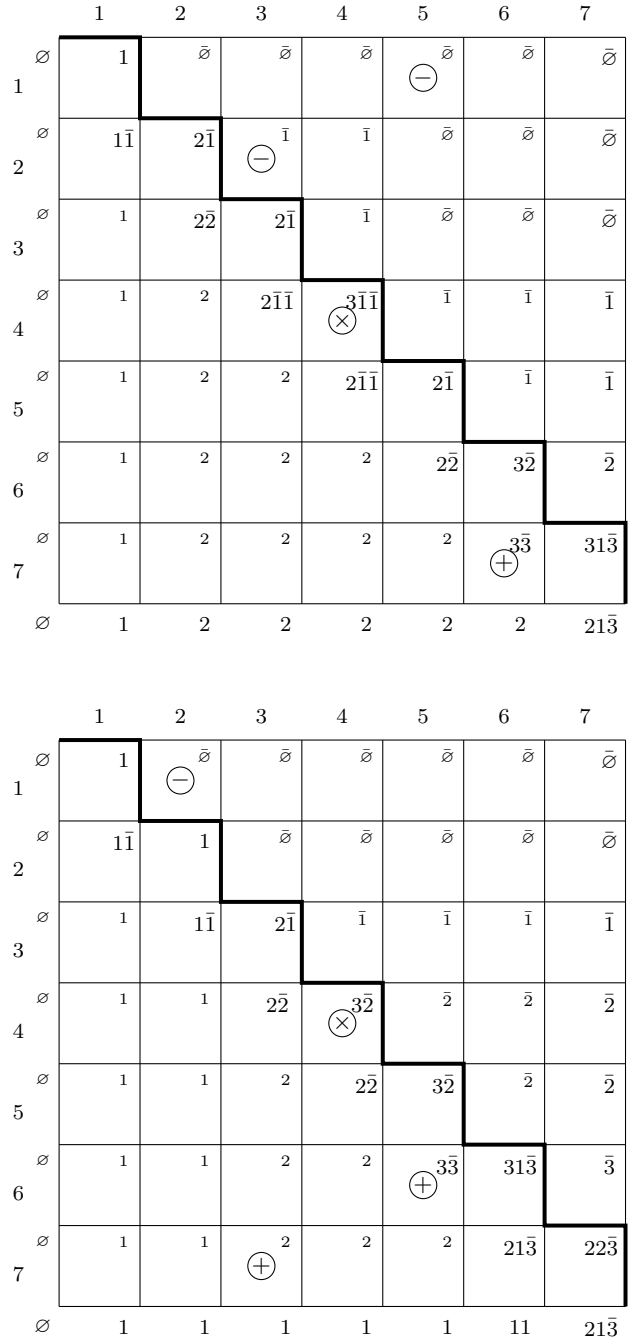


FIGURE 5. A pair of growth diagrams $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(\mathrm{s}_{1,7}\,\mathcal{A})$, with $\mathcal{A} = (\varnothing, 1, 1\bar{1}, 2\bar{1}, 2\bar{2}, 2\bar{1}, 2\bar{1}\bar{1}, 3\bar{1}\bar{1}, 2\bar{1}\bar{1}, 2\bar{1}, 2\bar{2}, 3\bar{2}, 3\bar{3}, 31\bar{3}, 21\bar{3})$, illustrating Theorem 3.6.

Before proceeding to the proof, let us remark that this is not a trivial statement: it may well be that some staircases in the intermediate row $\acute{\mathrm{p}}\mathrm{r}\,\mathcal{A}$ have more than m nonzero parts.

Proof. It suffices to consider the case $m = n - 1$. We show inductively that the statement is true for every square of staircases in diagram (3)

$$
\begin{array}{ccccc}
\lambda = \mu^{2i-2} & \xrightarrow{+} & \alpha = \mu^{2i-1} & \xrightarrow{-} & \nu = \mu^{2i} \\
-\uparrow & & -\uparrow & & -\uparrow \\
\beta = \acute{\mu}^{2i-3} & \xrightarrow{+} & \varepsilon = \acute{\mu}^{2i-2} & \xrightarrow{-} & \gamma = \acute{\mu}^{2i-1} \\
+\uparrow & & +\uparrow & & +\uparrow \\
\kappa = \hat{\mu}^{2i-4} & \xrightarrow{+} & \delta = \hat{\mu}^{2i-3} & \xrightarrow{-} & \mu = \hat{\mu}^{2i-2}
\end{array}
\tag{6}
$$

where a + between two staircases indicates that a unit vector is added to the staircase on the left (respectively, in the lower row) to obtain the staircase on the right (respectively, in the upper row).

By assumption, all staircases in the top and bottom row contain at least one zero entry. For such a staircase $\rho \in \mathbb{Z}^n$, let $\tilde{\rho} \in \mathbb{Z}^{n-1}$ be the staircase obtained from ρ by removing a zero entry. If ρ does not contain a zero, it must contain an entry 1 (say, at position i), followed by a negative entry. In this case, $\tilde{\rho} \in \mathbb{Z}^{n-1}$ is obtained from ρ by removing ρ_i and adding 1 to ρ_{i+1}.

With this notation, we have to show the following four equalities

(a) $\tilde{\varepsilon} = \mathrm{dom}_{\mathfrak{S}_{n-1}}(\tilde{\beta} + \tilde{\alpha} - \tilde{\lambda})$,
(b) $\tilde{\gamma} = \mathrm{dom}_{\mathfrak{S}_{n-1}}(\tilde{\varepsilon} + \tilde{\nu} - \tilde{\alpha})$,
(c) $\tilde{\delta} = \mathrm{dom}_{\mathfrak{S}_{n-1}}(\tilde{\kappa} + \tilde{\varepsilon} - \tilde{\beta})$, and
(d) $\tilde{\mu} = \mathrm{dom}_{\mathfrak{S}_{n-1}}(\tilde{\delta} + \tilde{\gamma} - \tilde{\varepsilon})$.

Let us first reduce to the case where at least one of the staircases involved does not contain a zero. Consider a square of staircases

$$
\begin{array}{ccc}
\beta = \alpha \pm e_i & \longrightarrow & \delta = \alpha \pm e_i \pm e_j \\
\uparrow & & \uparrow \\
\alpha & \longrightarrow & \gamma = \mathrm{dom}_{\mathfrak{S}_n}(\alpha \pm e_j)
\end{array}
$$

where all of α, β, γ and δ contain a zero. We first show that there is an index $k \notin \{i, j\}$ such that $\alpha_k = \beta_k = \delta_k = 0$. Suppose on the contrary that $\alpha_k \neq 0$ for all $k \notin \{i, j\}$. Then, since β contains a zero, we have $i \neq j$. Furthermore, we have

$$
\begin{array}{lll}
\alpha_i = 0 & \text{or } \alpha_j = 0, & \text{and} \\
\alpha_i = \mp 1 & \text{or } \alpha_j = 0, & \text{and} \\
\alpha_i = 0 & \text{or } \alpha_j = \mp 1, & \text{and} \\
\alpha_i = \mp 1 & \text{or } \alpha_j = \mp 1 &
\end{array}
$$

because α, β, γ and δ contain a zero, respectively. However, this set of equations admits no solution. Thus, there must be a further zero in α and therefore also in β, γ and δ. From this it follows that $\tilde{\gamma} = \mathrm{dom}_{\mathfrak{S}_{n-1}}(\tilde{\alpha} + \tilde{\delta} - \tilde{\beta})$.

Returning to the square in (6), we show that ε contains a zero entry if β or γ do. Suppose on the contrary that ε does not contain a zero entry. Then $\varepsilon = \beta + e_i$, where i is the position of the (only) zero in β. Moreover, we have $\alpha = \beta$, because there is only one way to obtain a zero entry in α by subtracting a unit vector. Thus,

$$\lambda = \mathrm{dom}_{\mathfrak{S}_n}(\beta + \alpha - \varepsilon) = \mathrm{dom}_{\mathfrak{S}_n}(\beta - e_i) = \beta - e_i,$$

which implies that λ does not contain a zero entry, contradicting our assumption. Similarly, if γ contains a zero at position i, we have $\varepsilon = \gamma + e_i$, $\gamma = \delta$ and $\mu = \mathrm{dom}_{\mathfrak{S}_n}(\gamma - e_i)$, a contradiction.

There remain three different cases:

β contains a zero, but γ does not.

We have to show Equations (b) and (d). Let $\alpha = \varepsilon - e_i$ and $\nu = \varepsilon - e_i - e_j$. Then $\gamma = \mathrm{dom}_{\mathfrak{S}_n}(\varepsilon - e_j)$. Since, by the foregoing, ε contains a zero, we have $\varepsilon_j = 0$. Since α also has a zero we have $i \neq j$. Since ν has a zero, $\varepsilon_i = 1$. Because γ has no zero, $\mu = \nu$. Together with the fact that δ has a zero, this implies that $\delta = \varepsilon - e_i$. The equations can now be checked directly.

β contains no zero, but γ does.

We have to show Equations (a) and (c). Let $\lambda = \beta - e_i$ and $\alpha = \beta - e_i + e_j$. Then $\varepsilon = \mathrm{dom}_{\mathfrak{S}_n}(\beta + e_j)$. Since β has no zero, but, by the foregoing, ε does, we have $\beta_j = \bar{1}$. Since λ has a zero, $\beta_i = 1$, and thus $i \neq j$. Because β has no zero, $\kappa = \lambda$. Again, the equations can now be checked directly.

None of β, ϵ and γ contain a zero.

In this case, $\kappa = \lambda$, $\delta = \alpha$ and $\mu = \nu$. Let $\lambda = \beta - e_i$, $\alpha = \beta - e_i + e_j$. Then $\varepsilon = \mathrm{dom}_{\mathfrak{S}_n}(\beta + e_j)$. Thus $\beta_j \neq \bar{1}$, $\beta_i = 1$, $\beta_{i+1} \leqslant \bar{1}$ and, because $\alpha \neq \beta$, we have $i \neq j$. Because α and β are staircases, $\beta + e_j$ has in fact decreasing entries and $\varepsilon = \beta + e_j$. Thus, $\alpha = \varepsilon - e_i$, $\nu = \varepsilon - e_i - e_k$ and $\gamma = \mathrm{dom}_{\mathfrak{S}_n}(\varepsilon - e_k)$. Again, because ε and ν are staircases, $\varepsilon - e_k$ has decreasing entries and $\gamma = \varepsilon - e_k$. Thus, the equations can now be checked directly. □

6.2. Growth diagrams for staircase tableaux. In this section we set up the notation used in the remaining sections. In particular, we slightly modify and generalise the definition of $\mathcal{G}(\mathcal{A})$ from Section 5.2.

Definition 6.2. *For a pair of partitions $\mu = (\mu_+, \mu_-)$, the partition μ_+ is the* positive part *and the partition μ_- is the* negative part. *Given an integer n not smaller than the sum of the lengths of the two partitions, $[\mu_+, \mu_-]_n$ is the staircase*

$$(\mu_{+,0}, \mu_{+,1}, \ldots, 0, \ldots, 0, \ldots, -\mu_{-,1}, -\mu_{-,0}).$$

A staircase tableau *is a sequence of staircases $\mathcal{A} = (\mu^0, \mu^1, \ldots, \mu^r)$ such that μ^i and μ^{i+1} differ by a unit vector for $0 \leqslant i < r$. If $\mu^0 = \varnothing$ the tableau is* straight, *otherwise it is* skew. *Unless explicitly stated, we consider only straight staircase tableaux.*

The extent[(2)] *$\mathcal{E}(\mu)$ of a staircase $\mu = [\mu_+, \mu_-]_n$ is the number of nonzero entries in μ. Put differently, the extent is the sum of the lengths of the partitions μ_+ and μ_-. The* extent *of a staircase tableau is the maximal extent of its staircases.*

Given a staircase tableau we can create a growth diagram similar to the procedure used in Section 5.2. However, it will be convenient to label *all* corners of the cells with staircases, instead of labelling the corners which are not on the main diagonal or first superdiagonal with a partition instead of a staircase.

Definition 6.3. *The growth diagram $\mathcal{G}(\mathcal{A})$ corresponding to a (straight) staircase tableau $\mathcal{A}$ is obtained in analogy to Definition* 5.3*: label the top-left corner with the staircase μ^0. If μ^{i+1} is obtained from μ^i by adding (respectively subtracting) a unit vector, μ^{i+1} labels the corner to the right of (respectively below) the corner labelled μ^i.* All *the remaining corners of $\mathcal{G}(\mathcal{A})$ are then labelled with staircases as follows. The positive parts on the corners to the left and below the path defined by the staircase*

(2) It might be more logical to use 'height' for the extent of a staircase, and 'length' for the number n. However, Stembridge defines the height of a staircase as the number n. We therefore avoid the words 'length' and 'height' in the context of staircases altogether.

tableau are obtained by applying the backward rule, whereas the forward rule determines the positive parts on the remaining corners. The negative parts are computed similarly.

Alternatively, we can also create a growth diagram given a partial filling and two partial standard Young tableaux.

Definition 6.4. *A* partial filling *ϕ is a rectangular array of cells, where every row and every column contains at most one cell with a cross.*

Let ϕ be a partial filling having crosses in all rows except $\mathcal{R}$ (counted from the top), and in all columns except $\mathcal{C}$ (counted from the left). Let P and Q be partial standard Young tableaux having entries $\mathcal{R}$ and $\mathcal{C}$ respectively. Then the growth diagram $\mathcal{G}(\phi, P, Q)$ is obtained as follows. The sequence of partitions corresponding to Q (respectively P) determines the positive (respectively negative) parts of the staircases on the bottom (respectively right) border. The remaining positive and negative parts are computed using the forward rule.

If ϕ contains precisely one cross in every row and every column, we abbreviate $\mathcal{G}(\phi, \varnothing, \varnothing)$ to $\mathcal{G}(\phi)$.

Finally, any growth diagram in the sense above can be decomposed into two classical growth diagrams, where all corners are labelled by partitions.

Definition 6.5. *$\mathcal{G}_+$ (respectively $\mathcal{G}_-$) denotes the (classical) growth diagrams obtained by ignoring the negative (respectively positive) parts of the staircases labelling the corners of a growth diagram $\mathcal{G}$.*

Remark 6.6. The classical growth diagram associated to a (partial) filling ϕ is precisely $\mathcal{G}_+(\phi, Q)$.

Remark 6.7. Two horizontally adjacent shapes in $\mathcal{G}_+(\phi, Q)$ differ if and only if there is no cross above in this column. Two horizontally adjacent shapes in $\mathcal{G}_-(\phi, Q)$ differ if and only if there is a cross above in this column.

Remark 6.8. Transposing a filling ϕ is equivalent to interchanging $\mathcal{G}_+(\phi)$ and $\mathcal{G}_-(\phi)$.

Finally, we introduce the operations on fillings we want to relate to promotion.

Definition 6.9. *Let ϕ be a filling of a square grid. The* column rotation $\operatorname{crot}\phi$ *(respectively,* row rotation $\operatorname{rrot}\phi$*) of the filling ϕ is obtained from ϕ by removing the first column (respectively, row) and appending it at the right (respectively, bottom).*

The rotation $\operatorname{rot}\phi$ *of a filling ϕ is* $\operatorname{crot}\operatorname{rrot}\phi$.

6.3. Promotion and evacuation of alternating tableaux. In this section we prove Theorems 3.6 and 3.7, with the exception of the case $n = 2$.

Let us first recall a classical fact concerning the effect of removing the first column of a filling on the growth diagram in terms of Schützenberger's jeu de taquin.

Proposition 6.10 ([21, A 1.2.10]). *Consider the classical growth diagrams $\mathcal{G}$ and $\acute{\mathcal{G}}$ for the partial fillings ϕ and $\acute{\phi}$, where $\acute{\phi}$ is obtained from ϕ by deleting its first column. Let Q and $\acute{Q}$ be the partial standard Young tableaux corresponding to the sequence of partitions on the top borders of the growth diagrams $\mathcal{G}$ and $\acute{\mathcal{G}}$. Then $\acute{Q} = \operatorname{jdt} Q$, the tableau obtained by applying Schützenberger's jeu de taquin to Q.*

The following central result connects the local rule for the symmetric group with column rotation, the operation of moving the first letter of a permutation to the end.

Theorem 6.11. *Let ϕ be a filling of an $r \times r$ square grid having exactly one cross in every row and in every column. Let λ and ν be two adjacent staircases in $\mathcal{G}(\phi)$,*

not on the left border of $\mathcal{G}(\phi)$, and λ being to the left of ν or above ν. Finally, let κ and μ be the two corresponding staircases in $\mathcal{G}(\operatorname{crot}\phi)$, that is, the column index of κ in $\mathcal{G}(\operatorname{crot}\phi)$ is one less than the column index of λ in $\mathcal{G}(\phi)$. Then, provided that $n \geqslant \max(\mathcal{E}(\kappa), \mathcal{E}(\lambda), \mathcal{E}(\mu), \mathcal{E}(\nu))$, we have $\mu = \operatorname{dom}_{\mathfrak{S}_n}(\kappa + \nu - \lambda)$.

Because the filling and the staircases of a growth diagram determine each other uniquely, we immediately obtain the following corollary.

Corollary 6.12. *Let ϕ be a filling of an $r \times r$ square grid having exactly one cross in every row and in every column. Suppose that the staircases in $\mathcal{A} = (1 = \mu^1, \dots, \mu^{2r} = \varnothing)$ label a sequence of adjacent corners from the corner just to the right of the top-left corner to the bottom-right corner of $\mathcal{G}(\phi)$. Furthermore, suppose that the staircases $\acute{\mathcal{A}} = (\varnothing = \acute{\mu}^0, \dots, \acute{\mu}^{2r} = \varnothing)$ satisfy $\acute{\mu}^i = \operatorname{dom}_{\mathfrak{S}_n}(\acute{\mu}^{i-1} + \mu^{i+1} - \mu^i)$ for $i \leqslant 2r-1$. Then, provided that $n \geqslant \max(\mathcal{E}(\mathcal{A}), \mathcal{E}(\acute{\mathcal{A}}))$, the filling of $\mathcal{G}(\acute{\mathcal{A}})$ is $\operatorname{crot}\phi$.*

We remark that Proposition 6.10, restricted to permutations, is a special case of Theorem 6.11. More precisely, it is obtained by considering the staircase tableau $(1 = \mu^1, \dots, \mu^{2r} = \varnothing)$ consisting of the partitions labelling the corners along the top and the right border of a classical growth diagram, with the empty shape in the top-left corner removed.

It is not hard to extend the theorem to partial fillings; the statement is completely analogous. Its proof proceeds by extending the partial filling to a permutation. However, it turns out to be more convenient to deduce the statements for staircase tableaux of non-empty shape from the corresponding statements for staircase tableaux of empty shape directly.

Proof of Theorem 6.11. Local rules for the positive and the negative parts.

Let us first determine certain local rules satisfied separately by the positive and negative parts of the staircases κ, λ, μ and ν. A summary of the various cases is displayed in Figure 6, where the rules we verify are displayed below the corresponding diagrams. In the following, addition and subtraction of integer partitions is defined by interpreting them as vectors in $\mathbb{Z}^n$.

First case, λ left of ν.

Let $Q = (\varnothing = \mu_0, \mu_1, \dots, \mu_{s-1} = \lambda_+, \mu_s = \nu_+)$ be the partial standard Young tableau corresponding to the sequence of partitions in $\mathcal{G}_+(\phi)$ on the same row as λ and ν, beginning at the left border. Let $\acute{Q} = (\varnothing = \acute{\mu}_0, \acute{\mu}_1, \dots, \acute{\mu}_{s-2} = \kappa_+, \acute{\mu}_{s-1} = \mu_+)$ be the corresponding partial standard Young tableau in $\mathcal{G}_+(\operatorname{crot}\phi)$.

Suppose there is a cross in ϕ in the first column in a row below ν, as in Figure 6(a). Then, by Proposition 6.10, $\acute{Q} = \operatorname{jdt} Q$. This implies that the partitions $\mu_{s-1}, \mu_s, \acute{\mu}_{s-2}, \acute{\mu}_{s-1}$ satisfy the local (growth diagram) rule $\acute{\mu}_{s-1} = \operatorname{sort}(\acute{\mu}_{s-2} + \mu_s - \mu_{s-1})$, that is, $\mu_+ = \operatorname{sort}(\kappa_+ + \nu_+ - \lambda_+)$. Moreover $\kappa_- = \lambda_-$ and $\nu_- = \mu_-$ because the growth for the negative parts of the staircases is from the top-right to the bottom-left.

If there is a cross in the first column in a row above ν, as in Figure 6(b), we reason in a very similar way. In this case $\kappa_+ = \lambda_+$ and $\nu_+ = \mu_+$. For the negative parts of the staircases we consider the partial standard Young tableaux Q and $\acute{Q}$ corresponding to the sequences of partitions beginning at the *right* border of $\mathcal{G}_-(\phi)$ and $\mathcal{G}_-(\operatorname{crot}\phi)$ respectively. We then have $Q = \operatorname{jdt}\acute{Q}$ and conclude $\lambda_- = \operatorname{sort}(\kappa_- + \nu_- - \mu_-)$ as before, using the symmetry of the local rule, see Remark 4.14.

Second case, λ above ν.

Depending on the position of the cross in the first column there are three slightly different cases, as illustrated in Figure 6(c), (d) and (e).

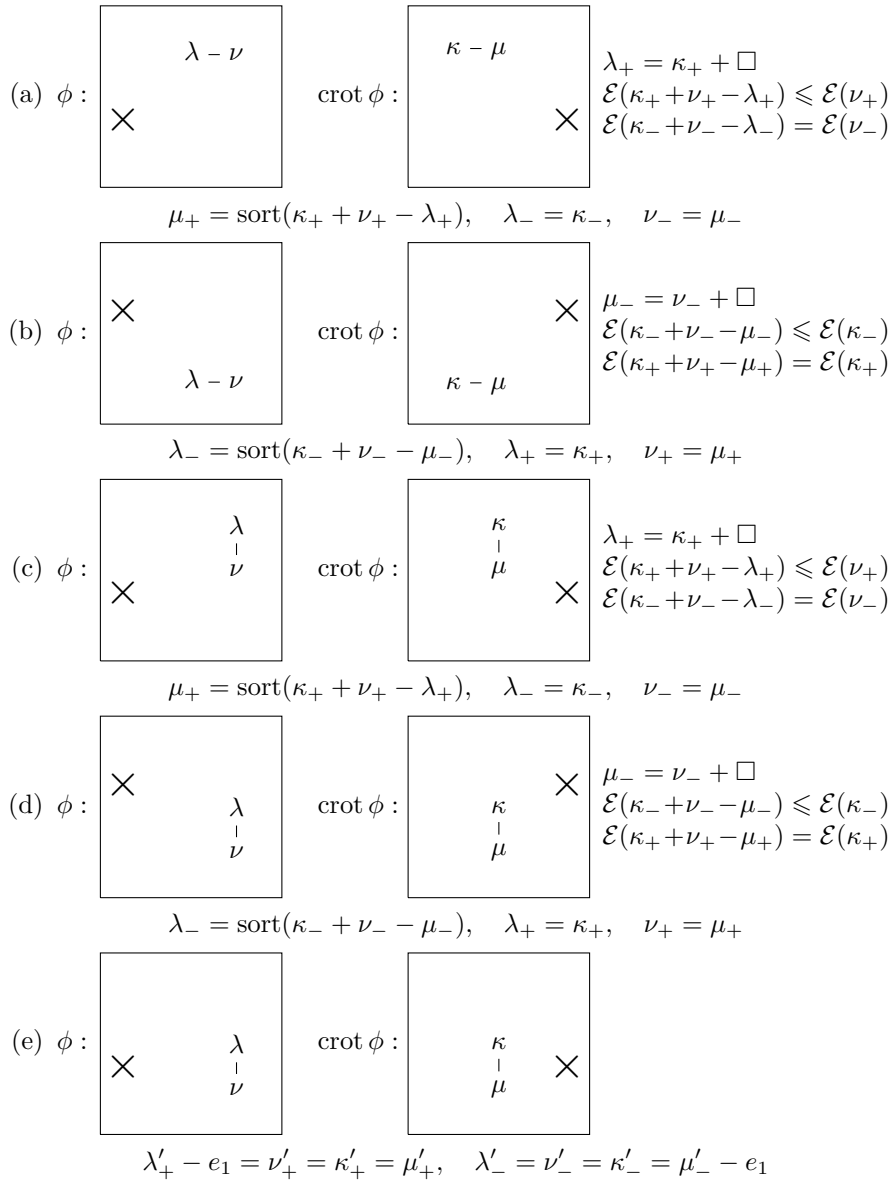


FIGURE 6. The cases considered in the proof of Theorem 6.11.

Recall that the partitions on the right border of a (classical) growth diagram corresponding to the right to left reversal of a filling ψ are obtained by transposing the partitions on the right border of the (classical) growth diagram corresponding to ψ. Consider now the filling ψ below and to the left of λ, and let $\acute{\psi}$ be the filling to the left and below κ. Note that the reversal of ψ is obtained from the reversal of $\acute{\psi}$ by appending the first column of ψ to the right. We thus obtain that the transposes of the positive parts of the staircases κ, μ, λ and ν satisfy the local (growth diagram) rule.

The relation between the negative parts of the staircases κ, μ, λ and ν is obtained in a very similar way by considering the fillings above and to the right of ν and μ.

Bounding the extent and deducing the local rule.

We now show $\mu = \mathrm{dom}_{\mathfrak{S}_n}(\kappa+\nu-\lambda)$, provided $n \geqslant \max(\mathcal{E}(\kappa), \mathcal{E}(\lambda), \mathcal{E}(\mu), \mathcal{E}(\nu))$. To do so, we extend the notion of extent to arbitrary vectors with all entries non-negative or all entries non-positive: for a vector $\alpha_+ \in \mathbb{Z}^n_{\geqslant 0}$, the extent $\mathcal{E}(\alpha_+)$ is n minus the number of trailing zeros. Similarly, for a vector $\alpha_- \in \mathbb{Z}^n_{\leqslant 0}$, the extent $\mathcal{E}(\alpha_-)$ is n minus the number of leading zeros.

The case illustrated in Figure 6(e) follows by direct inspection. We thus only consider the remaining four cases. Because μ_+ and μ_- are obtained by sorting $\kappa_+ + \nu_+ - \lambda_+$ and $\kappa_- + \nu_- - \lambda_-$ respectively, the latter must have all entries non-negative. Similarly, also $\kappa_+ + \nu_+ - \mu_+$ and $\kappa_- + \nu_- - \mu_-$ have all entries non-negative, because λ_+ and λ_- are obtained by sorting these vectors, by the symmetry of the local rule, see Remark 4.14.

Suppose first that $n \geqslant \max(\mathcal{E}(\kappa), \mathcal{E}(\lambda), \mathcal{E}(\nu))$. Then

$$\mathrm{dom}_{\mathfrak{S}_n}(\kappa+\nu-\lambda) = \mathrm{dom}_{\mathfrak{S}_n}([\kappa_+,\kappa_-]_n + [\nu_+,\nu_-]_n - [\lambda_+,\lambda_-]_n) = \mathrm{dom}_{\mathfrak{S}_n}(\alpha_+ + \alpha_-),$$

where $\alpha_+ = [\kappa_+, \varnothing]_n + [\nu_+, \varnothing]_n - [\lambda_+, \varnothing]_n$ and $\alpha_- = [\varnothing, \kappa_-]_n + [\varnothing, \nu_-]_n - [\varnothing, \lambda_-]_n$. It remains to show that

$$\mathcal{E}(\alpha_+) + \mathcal{E}(\alpha_-) = \mathcal{E}\left(\kappa_+ + \nu_+ - \lambda_+\right) + \mathcal{E}\left(\kappa_- + \nu_- - \lambda_-\right) \leqslant n,$$

because then

$$\begin{aligned}\mathrm{dom}_{\mathfrak{S}_n}(\alpha_+ + \alpha_-) &= [\mathrm{sort}(\alpha_+), \mathrm{sort}(\alpha_-)]_n \\ &= [\mathrm{sort}(\kappa_+ + \nu_+ - \lambda_+), \mathrm{sort}(\kappa_- + \nu_- - \lambda_-)]_n = [\mu_+, \mu_-]_n = \mu.\end{aligned}$$

Similarly, suppose that $n \geqslant \max(\mathcal{E}(\kappa), \mathcal{E}(\mu), \mathcal{E}(\nu))$. In this case, reasoning as above, we have to show that $\mathcal{E}\left(\kappa_+ + \nu_+ - \mu_+\right) + \mathcal{E}\left(\kappa_- + \nu_- - \mu_-\right) \leqslant n$.

The first inequality is verified by inspection of Figure 6(a) and (c), whereas the second concerns Figure 6(b) and (d). Here we write, for example, $\lambda_+ = \kappa_+ + \square$ to indicate that the partition λ_+ is obtained from the partition κ_+ by adding a single cell, which implies the inequality for the extent. □

Definition 6.13. *Let $\mathcal{A} = (\varnothing = \mu^0, \mu^1, \ldots, \mu^{2r-1}, \mu^{2r} = \mu)$ be an alternating tableau. Then $\acute{\mathrm{p}}\mathrm{r}\,\mathcal{A} = (\varnothing = \acute{\mu}^0, \acute{\mu}^1, \ldots, \acute{\mu}^{2r-1}, \acute{\mu}^{2r} = \mu)$ is the staircase tableau obtained from $\mathcal{A}$ by setting $\acute{\mu}^1 = \mu^1 = 1$, and then applying the local rule (1) successively to μ^i, μ^{i+1}, and $\acute{\mu}^{i-1}$ to obtain $\acute{\mu}^i$ for $i \leqslant 2r-1$. Additionally, we set $\acute{\mu}^{2r} = \mu$.*

In other words, $\acute{\mathrm{p}}\mathrm{r}\,\mathcal{A}$ can be read off from the diagram for promotion as illustrated in diagram (3) beginning with the empty shape in the lower left corner, then following the second row, and terminating with the shape μ in the upper right corner.

Lemma 6.14.

(a) *Let $\mathcal{A}$ be a staircase tableau of empty shape and length r. Then the extent of $\mathcal{A}$ is at most r.*

(b) *Restricting to alternating tableaux, there is a single alternating tableau $\mathcal{A}_0$ of empty shape, length r and extent r. The filling ϕ_0 of its growth diagram $\mathcal{G}(\mathcal{A}_0)$ is invariant under rotation:* $\mathrm{rot}\,\phi_0 = \phi_0$.

(c) *Restricting further to alternating tableaux of even length, the only tableau $\mathcal{A}$ such that $\acute{\mathrm{p}}\mathrm{r}\mathcal{A}$ has extent r is $\mathcal{A}_0$.*

Proof. Statement (a) is trivial. To see statement (b), note that the unique length r alternating tableau of empty shape with maximal extent is, for $r = 2s+1$ odd,

$$\mathcal{A}_0 = (\varnothing, 1, 1\bar{1}, \ldots, 1^s\bar{1}^s, 1^{s+1}\bar{1}^s, 1^s\bar{1}^s, \ldots, 1\bar{1}, 1, \varnothing)$$

and, for $r = 2s$ even,

$$\mathcal{A}_0 = (\varnothing, 1, 1\bar{1}, \ldots, 1^s\bar{1}^s, 1^s\bar{1}^{s-1}, 1^{s-1}\bar{1}^{s-1}, \ldots, 1\bar{1}, 1, \varnothing).$$

In both cases the extent is r and the corresponding permutation is, in one line notation, $s+1, s+2, \ldots, r, 1, \ldots, s$. This permutation is invariant under rotation.

Similarly, to see statement (c), a staircase tableau $\acute{\mathrm{pr}}\mathcal{A}$ with extent r must have filling corresponding to the permutation $s, s+1, \ldots, r, 1, \ldots, s-1$, which is the filling $\mathrm{rrot}\,\phi$, and thus $\mathcal{A} = \mathcal{A}_0$. □

The first statement of Theorem 3.7, with the exception of the case $n = 2$ and the case $n = r - 1$, is a direct consequence of the following result. The special case of GL(2)-alternating tableaux and GL$(r-1)$-alternating tableaux will be considered in Section 6.4 below.

THEOREM 6.15. *Let $\mathcal{A}$ be a* GL(n)*-alternating tableau of length r and empty shape. Let ϕ be the filling of the growth diagram $\mathcal{G}(\mathcal{A})$. Let $\acute{\phi}$ and $\hat{\phi}$ be the fillings of the growth diagrams $\mathcal{G}(\acute{\mathrm{pr}}\,\mathcal{A})$ respectively $\mathcal{G}(\mathrm{pr}\,\mathcal{A})$.*

Then, for $n \geqslant r$, $\mathrm{rrot}\,\phi = \acute{\phi}$ *and* $\mathrm{crot}\,\acute{\phi} = \hat{\phi}$.

Proof. Let

$$\acute{\mathrm{pr}}\,\mathcal{A} = (\varnothing = \acute{\mu}^0, \acute{\mu}^1, \ldots, \acute{\mu}^{2r-1} = 1, \acute{\mu}^{2r} = \varnothing)$$

and let

$$\mathrm{pr}\,\mathcal{A} = (\varnothing = \hat{\mu}^0, \hat{\mu}^1, \ldots, \hat{\mu}^{2r-1} = 1, \hat{\mu}^{2r} = \varnothing).$$

Furthermore, let

$$\widetilde{\mathcal{A}} = (\varnothing = \widetilde{\mu}^0, \widetilde{\mu}^1, \ldots, \widetilde{\mu}^{2r-1} = \bar{1}, \widetilde{\mu}^{2r} = \varnothing)$$

be the staircase tableau obtained by setting $\widetilde{\mu}^0 = \varnothing$ and then applying the local rule (1) successively to $\acute{\mu}^i$, $\acute{\mu}^{i+1}$, and $\widetilde{\mu}^{i-1}$ to obtain $\widetilde{\mu}^i$ for $i \leqslant 2r-1$. Because of Lemma 6.14 and the assumption $n \geqslant r$, Corollary 6.12 is applicable and implies that the filling $\widetilde{\phi}$ of the growth diagram $\mathcal{G}(\widetilde{\mathcal{A}})$ is $\mathrm{crot}\,\acute{\phi}$.

All staircases in $\widetilde{\mathcal{A}}$ except $\widetilde{\mu}^{2r-1}$ coincide with those of $\mathrm{pr}\,\mathcal{A}$. Because $\acute{\mu}^{2r-1} = 1$, $\acute{\mu}^{2r} = \varnothing$ and $\widetilde{\mu}^{2r-2}$ is either $\varnothing$ or $1\bar{1}$, we have $\widetilde{\mu}^{2r-1} = \bar{1}$. However, since $\widetilde{\phi}$ and $\hat{\phi}$ correspond to permutations and the first $r-1$ columns of these fillings are the same, we conclude that $\widetilde{\phi}$ equals $\hat{\phi}$.

Because of the symmetry of the local rules pointed out in Remark 4.14 and because $\acute{\mu}^{2r-1} = 1$, we can apply the same reasoning replacing $\acute{\mathrm{pr}}\,\mathcal{A}$ and $\mathrm{pr}\,\mathcal{A}$ with the reversal of $\acute{\mathrm{pr}}\,\mathcal{A}$ and the reversal of $\mathcal{A}$. Clearly, the filling corresponding to the reversal of a tableau is obtained by flipping the original filling over the diagonal from the bottom-left to the top-right. In the process, column rotation is replaced by row rotation, which implies that $\mathrm{rrot}(\phi) = \acute{\phi}$. □

COROLLARY 6.16. *In the setting of Theorem* 6.15*, if n is odd it is sufficient to require $n \geqslant r - 1$.*

Proof. Let $\mathcal{A}$ be a GL(n)-alternating tableau of length $r = 2s$, with $n \geqslant r$. Let ϕ be the filling of $\mathcal{G}(\mathcal{A})$. Then, combining Lemma 6.14 and Theorem 6.15 we obtain

$$\mathcal{E}(\mathcal{A}) = r \Leftrightarrow \mathcal{E}(\acute{\mathrm{pr}}\mathcal{A}) = r \Leftrightarrow \mathcal{E}(\mathrm{pr}\,\mathcal{A}) = r.$$

By contraposition, $\mathcal{E}(\mathcal{A}) < r \Leftrightarrow \mathcal{E}(\acute{\mathrm{pr}}\mathcal{A}) < r \Leftrightarrow \mathcal{E}(\mathrm{pr}\,\mathcal{A}) < r$. Thus, the claim follows using the proof of Theorem 6.15, taking into account that Corollary 6.12 is now applicable even with $n \geqslant r - 1$. □

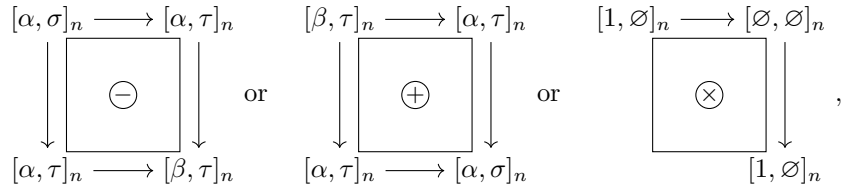
Finally, we can conclude one part of Theorem 3.7. Note that the case of odd n is also covered by the previous corollary.

Corollary 6.17. *Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau of length r and empty shape. Then, for $n \geqslant r-1$, $\operatorname{rot}\mathcal{P}(\mathcal{A}) = \mathcal{P}(\operatorname{pr}\mathcal{A})$.*

Proof. This is a consequence of Lemma 6.14 and Theorem 6.1. □

We now introduce a different way to obtain the filling of $\mathcal{G}(\mathcal{A})$. This construction will also shed some additional light on the relationship between the local rule (1) and those in Figure 3.

Consider the evacuation diagram for obtaining the evacuation as illustrated in Figure 2. We construct a filling of the cells surrounded by three or four staircases using the symbols $\ominus$, $\oplus$ and $\otimes$ as follows:



where β (respectively σ) is obtained from α (respectively τ) by adding a cell to the first column of the Ferrers diagram of the partition. All other cells remain empty.

The following lemma is the main building block in establishing the connection between the filling of $\mathcal{G}(\mathcal{A})$ and the decorated evacuation diagram.

Lemma 6.18. *Let $\mathcal{A} = (\varnothing = \mu^0, \dots, \mu^{2r} = \varnothing)$ be an alternating tableau of empty shape. Let $\operatorname{pr}\mathcal{A} = (\varnothing = \acute{\mu}^0, \acute{\mu}^1, \dots, \acute{\mu}^{2r-1}, \acute{\mu}^{2r} = \varnothing)$ be as in Definition 6.13 and let $\operatorname{pr}\mathcal{A} = (\varnothing = \hat{\mu}^0, \dots, \hat{\mu}^{2r} = \varnothing)$ be the promotion of $\mathcal{A}$. Suppose that the filling of $\mathcal{G}(\mathcal{A})$ has a cross (that is, a $\ominus$, $\oplus$ or $\otimes$) in column $\ell > 1$ of the first row and in row $k > 1$ of the first column. Then, for even $n \geqslant r$ and for odd $n \geqslant r-1$, we have*

(a) $\mu^j_+ = \acute{\mu}^{j-1}_+$ *for* $2 \leqslant j \leqslant 2\ell - 2$,
(b) $\mu^j_- = \acute{\mu}^{j-1}_-$ *for* $j > 2\ell - 2$,
(c) $\mu^{2\ell-2}_+ = \mu^{2\ell-1}_+ = \acute{\mu}^{2\ell-3}_+$, *and* $\acute{\mu}^{2\ell-2}_+$ *is obtained from these by adding a cell to the first column. The cell labelled with these four staircases contains a* $\ominus$.

Similarly,

(a′) $\acute{\mu}^j_- = \hat{\mu}^{j-1}_-$ *for* $1 \leqslant j \leqslant 2k - 2$,
(b′) $\acute{\mu}^j_+ = \hat{\mu}^{j-1}_+$ *for* $j > 2k - 2$,
(c′) $\acute{\mu}^{2k-1}_+ = \hat{\mu}^{2k-2}_+ = \hat{\mu}^{2k-3}_+$, *and* $\acute{\mu}^{2k-2}_+$ *is obtained from these by adding a cell to the first column. The cell labelled with these four staircases contains a* $\oplus$.

Finally, suppose that there is a cross in the top-left cell, that is $k = \ell = 1$. Then

(f) $\mu^1 = 1, \mu^2 = \varnothing$ *and* $\acute{\mu}^1 = 1$. *The cell labelled with these four staircases contains a* $\otimes$.
(f′) $\mu^j = \acute{\mu}^{j-1} - e_1 = \hat{\mu}^{j-2}$ *for all* $2 \leqslant j \leqslant 2r$.

Proof. Consider a square of four adjacent staircases in the diagram for computing the promotion of an alternating tableau below:

$$
\begin{array}{ccccccccc}
 & & & \mu^2 & \dots & \mu^{2\ell-2} & \mu^{2\ell-1} & \dots\dots\dots\dots\dots & \mu^{2r} = \varnothing \\
 & & & & & & \ominus & & \\
 & & \acute{\mu}^1 & \dots & \acute{\mu}^{2\ell-3} & \acute{\mu}^{2\ell-2} & \dots & \acute{\mu}^{2k-2}\ \acute{\mu}^{2k-1} & \dots\ \acute{\mu}^{2r-1} \\
 & & & & & & & \oplus & \\
\varnothing = \hat{\mu}^0 & \dots\dots\dots\dots\dots & & & & & & \hat{\mu}^{2k-3}\ \hat{\mu}^{2k-2} & \dots\ \hat{\mu}^{2r-2}
\end{array}
\tag{7}
$$

By definition, these satisfy the local rule, as required by Theorem 3.3. By Corollary 6.16, Theorem 6.11 is applicable with the given bounds for n. The equalities for the staircases in the second and third row are precisely the equalities listed below the illustrations in Figure 6: cases (a) and (c) there describe the situation to the left of $\oplus$, case (e) describes the situation at $\oplus$ and cases (b) and (d) describe the situation to the right of $\oplus$.

The equalities for the staircases in the first and second row can be obtained as in the last paragraph of the proof of Theorem 6.15. □

By successively applying Lemma 6.18 we obtain the following result for the evacuation diagram.

COROLLARY 6.19. *Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau of empty shape and length r with corresponding growth diagram $\mathcal{G}(\mathcal{A})$ and filling ϕ. Suppose that $n \geqslant r$ if n is even and $n \geqslant r-1$ if n is odd. Consider the evacuation diagram with filling obtained as above. A $\ominus$ appears only in odd columns and odd rows, a $\oplus$ appears only in even columns and even rows and a $\otimes$ appears only in even columns and odd rows. Moreover (i,j) is the position of a cell with a cross in ϕ if and only if one of the following cases holds.*

- *$i < j$ and there is a $\ominus$ in row $2i-1$ and column $2j-1$ in the evacuation diagram.*
- *$i > j$ and there is a $\oplus$ in row $2j$ and column $2i$.*
- *There is a $\otimes$ in row $2i-1$ and column $2j$. Then we also obtain $i = j$.*

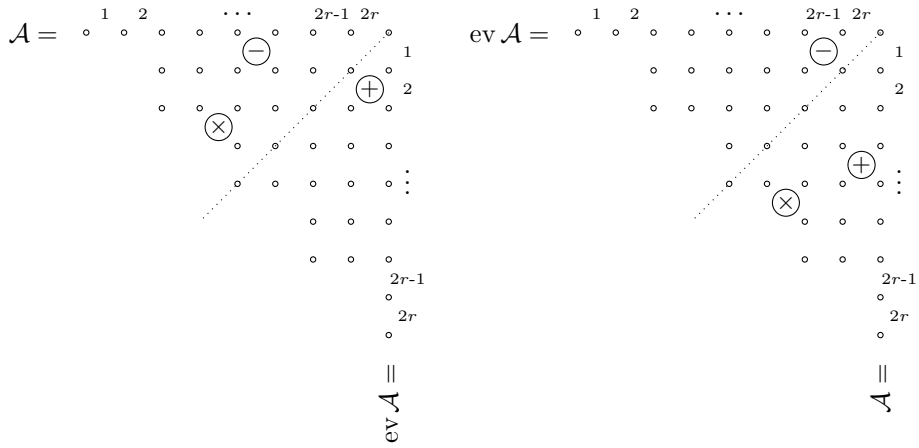


FIGURE 7. The symmetry of the evacuation diagram.

By the symmetry of the local rules the evacuation diagram for $\mathrm{ev}\,\mathcal{A}$ is obtained from the evacuation diagram for $\mathcal{A}$ by mirroring it along the diagonal and interchanging $\oplus$ and $\ominus$. The cell (i,j) is interchanged with the cell $(2r+1-j, 2r+1-i)$. This yields the part of Theorem 3.7 concerning evacuation, see Figure 7 for an illustration.

THEOREM 6.20. *Let $\mathcal{A} = (\varnothing = \mu^0, \mu^1, \ldots, \mu^{2r-1}, \mu^{2r} = \varnothing)$ be an alternating tableau. Suppose that $n \geqslant r$ if n is even and $n \geqslant r-1$ if n is odd. Let ϕ be the filling of the growth diagram $\mathcal{G}(\mathcal{A})$. Then the filling of $\mathcal{G}(\mathrm{ev}\,\mathcal{A})$ is obtained by rotating ϕ by $180°$.*

Proof. Let $\phi_{\mathcal{A}} = \phi$, respectively $\phi_{\mathrm{ev}\,\mathcal{A}}$, be the fillings of the growth diagrams $\mathcal{G}(\mathcal{A})$, respectively $\mathcal{G}(\mathrm{ev}\,\mathcal{A})$. Let (i,j) be the position of a cell with a cross in the filling $\phi_{\mathcal{A}}$. Then, according to Corollary 6.19:

(1) if $i < j$, there is a $\ominus$ in the evacuation diagram of $\mathcal{A}$ in $(2i-1, 2j-1)$. Thus there is a $\oplus$ in the evacuation diagram of $\operatorname{ev}\mathcal{A}$ in $(2r-2j+2, 2r-2i+2)$ and therefore there is a cross in $(r+1-i, r+1-j)$ in $\phi_{\operatorname{ev}\mathcal{A}}$.
(2) if $j < i$, there is a $\oplus$ in the evacuation diagram of $\mathcal{A}$ in $(2j, 2i)$. Thus there is a $\ominus$ in the evacuation diagram of $\operatorname{ev}\mathcal{A}$ in $(2r+1-2i, 2r+1-2j)$ and therefore there is a cross in $(r+1-i, r+1-j)$ in $\phi_{\operatorname{ev}\mathcal{A}}$.
(3) if $i = j$, then there is a $\otimes$ in the evacuation diagram of $\mathcal{A}$ in $(2i-1, 2i)$. Thus there is a $\otimes$ in the evacuation diagram of $\operatorname{ev}\mathcal{A}$ in $(2r+1-2i, 2r+2-2i)$ and therefore there is a cross in $(r+1-i, r+1-j)$ in $\phi_{\operatorname{ev}\mathcal{A}}$. □

Proposition 6.21. *Consider the classical growth diagrams $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ for the partial fillings ϕ and $\operatorname{rc}\phi$, where $\operatorname{rc}\phi$ is obtained by rotating ϕ by 180°. Let Q and $\widetilde{Q}$ (respectively P and $\widetilde{P}$) be the partial standard Young tableaux corresponding to the sequence of partitions on the top borders (respectively right borders) of the growth diagrams $\mathcal{G}$ and $\widetilde{\mathcal{G}}$. Then $\widetilde{Q} = \operatorname{ev} Q$ and $\widetilde{P} = \operatorname{ev} P$.*

We are now in the position to prove Theorem 3.6, which we reformulate as follows.

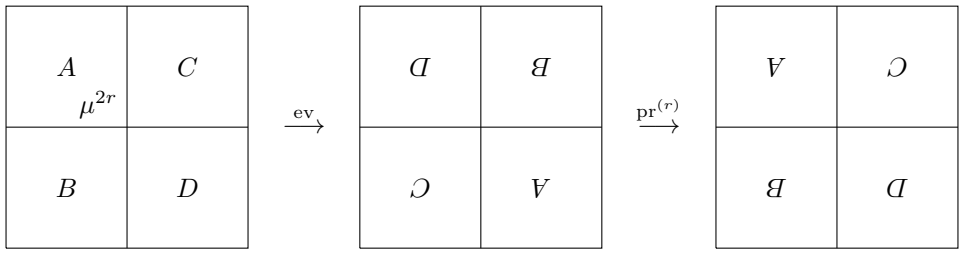
Theorem 6.22. *Let $\mathcal{A} = (\varnothing = \mu^0, \mu^1, \dots, \mu^{2r-1}, \mu^{2r} = \mu)$ be an alternating tableau of length $r \leqslant \lfloor \frac{n+1}{2} \rfloor$. Let ϕ be the filling of the growth diagram $\mathcal{G}(\mathcal{A})$. Then the sequence of partitions on the bottom (respectively right) border of $\mathcal{G}_+(\operatorname{ev}\mathcal{A})$ (respectively $\mathcal{G}_-(\operatorname{ev}\mathcal{A})$) is obtained by evacuating the sequence of partitions on the bottom (respectively right) border of $\mathcal{G}_+(\mathcal{A})$ (respectively $\mathcal{G}_-(\mathcal{A})$). Moreover, the filling of $\mathcal{G}(\operatorname{ev}\mathcal{A})$ is obtained by rotating ϕ by 180°.*

Proof. We begin by extending $\mathcal{A}$ to an alternating tableau of empty shape $\tilde{\mathcal{A}} = (\varnothing = \tilde{\mu}^0, \dots, \tilde{\mu}^{2(r+r)} = \varnothing)$, such that $\tilde{\mu}^i = \mu^i$ for $i \leqslant r$, by appending the reversal of $\mathcal{A}$. Let $\tilde{\phi}$ be the filling of $\mathcal{G}(\tilde{\mathcal{A}})$, which we divide into four parts, as illustrated in the left-most diagram below. Filling A is the filling corresponding to $\mathcal{A}$, filling B is the part below and to the left of μ^{2r}, filling C is the part above and to the right of μ^{2r} and filling D is the part below and to the right of μ^{2r}.

By the symmetry of the local rules and the evacuation diagram as illustrated in Figure 2 we see that $\operatorname{ev}\mathcal{A}$ coincides with the first $2r+1$ staircases of $\operatorname{pr}^{(r)}(\operatorname{ev}\tilde{\mathcal{A}})$, where $\operatorname{pr}^{(r)}$ denotes $\underbrace{\operatorname{pr}\circ\operatorname{pr}\circ\cdots\circ\operatorname{pr}}_{r \text{ times}}$.

Let Q be the sequence of partitions on the bottom border of $\mathcal{G}_+(\mathcal{A})$. This sequence is also the sequence of partitions on the top border of the classical growth diagram with filling B.

The inequality $r \leqslant \lfloor \frac{n+1}{2} \rfloor$ implies that $n \geqslant 2r$ if n is even and $n \geqslant 2r-1$ if n is odd. Applying Theorem 6.20 and Theorem 6.15 we obtain the following picture:



Thus the sequence of partitions on the bottom border of $\mathcal{G}_+(\operatorname{ev}\mathcal{A})$ is the same as the sequence of partitions on the top border of the regular growth diagram with filling $\operatorname{rc} B$. By Proposition 6.21 we obtain the statement for the sequence of partitions on the bottom border.

The result for the right border follows using the same argument, replacing the filling A with the filling C. □

6.4. GL(2)-ALTERNATING TABLEAUX. To finish the proof of Theorem 3.7, it remains to consider the case $n = 2$.

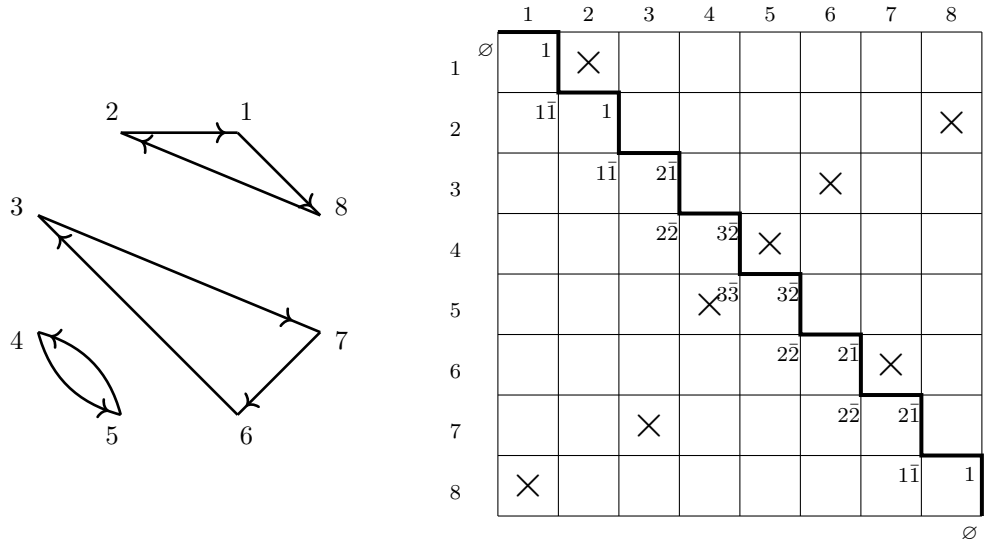


FIGURE 8. A noncrossing set partition corresponding to a GL(2)-alternating tableau.

LEMMA 6.23. *The map $\mathcal{P}$ restricts to a bijection between* GL(n)*-alternating tableaux of empty shape and length r, such that every staircase has at most two nonzero parts, and noncrossing set partitions on* $\{1, \ldots, r\}$.

Proof. For simplicity, suppose that $\mathcal{A}$ is a GL(2)-alternating tableau. Let π be the permutation corresponding to the filling associated with $\mathcal{A}$. We show that, when drawn as a chord diagram as in Figure 8, it is obtained from a noncrossing set partition by orienting the arcs delimiting the blocks clockwise, when the corners of the polygon are labelled counterclockwise.

We say that two arcs (i, π_i) and (j, π_j) in the chord diagram, with $i < k$, *cross*, if and only if the indices involved satisfy one of the following two inequalities:

$$i < j \leqslant \pi_i < \pi_j \qquad \text{or} \qquad \pi_i < \pi_j < i < j.$$

Let us remark that this is precisely Corteel's [4] notion of crossing in permutations.

It follows by direct inspection that the chord diagram corresponds to a noncrossing partition in the sense above if and only if no two arcs cross.

Moreover, a crossing of the first kind is the same as a pair of crosses in the rectangle below and to the left of the cell in row and column j of $\mathcal{G}(\mathcal{A})$, such that one cross is above and to the left of the other. Similarly, a crossing of the second kind is the same as a pair of crosses in the rectangle above and to the right of the cell in row and column i of $\mathcal{G}(\mathcal{A})$, such that one cross is above and to the left of the other.

By construction, a GL(2)-alternating tableau cannot contain a vector with both entries strictly positive or both entries strictly negative. Thus, such pairs of crosses may not occur. □

We can now prove another part of Theorem 3.7.

Theorem 6.24. *Let $n \leqslant 2$ and let $\mathcal{A}$ be a* GL(n)*-alternating tableau of empty shape. Then* $\operatorname{rot}\mathcal{P}(\mathcal{A}) = \mathcal{P}(\operatorname{pr}\mathcal{A})$.

Proof. Let r be the length of $\mathcal{A}$ and let $\hat{\mathcal{A}}$ be the GL(r)-alternating tableau obtained from $\mathcal{A}$ by inserting $r-n$ zeros into each staircase. Then, by Theorem 6.15, $\mathcal{P}(\operatorname{pr}\hat{\mathcal{A}}) = \operatorname{rot}\mathcal{P}(\hat{\mathcal{A}})$. By Lemma 6.23, the staircases in the alternating tableau corresponding to $\operatorname{rot}\mathcal{P}(\hat{\mathcal{A}})$ have at most two nonzero parts. Thus, the claim follows from Theorem 6.1. □

To finish the proof of Theorem 3.7, we show that the evacuation of a GL(2)-alternating tableaux of empty shape is just its reversal.

Theorem 6.25. *Let $\mathcal{A}$ be a* GL(2)*-alternating tableau of empty shape. Then* $\operatorname{ev}\mathcal{A}$ *is the reversal of $\mathcal{A}$.*

Proof. Let $\mathcal{A} = (\varnothing = \mu^0, \dots, \mu^{2r} = \varnothing)$ and let $\operatorname{ev}\mathcal{A} = \mathcal{A} = (\varnothing = \tilde{\mu}^0, \dots, \tilde{\mu}^{2r} = \varnothing)$. Note that $\tilde{\mu}^{2i}$ is the $2i$-th (counting from zero) staircase in $\operatorname{pr}^{(r-i)}\mathcal{A}$. Thus, its negative part is the same as the negation of the positive part of $\mu^{2(r-i)}$, because the fillings in the respective regions of the corresponding growth diagrams coincide. Because the negative part and the positive part of the even labelled staircases of a GL(2)-alternating tableau are equal, we conclude that $\mu^{2(r-i)} = \tilde{\mu}^{2i}$.

It remains to show that $\mu^{2(r-i)-1} = \tilde{\mu}^{2i+1}$. If $\tilde{\mu}^{2i} \neq \tilde{\mu}^{2(i+1)}$, the staircase $\tilde{\mu}^{2i+1}$ is uniquely determined. Otherwise, if $\tilde{\mu}^{2i} = \tilde{\mu}^{2(i+1)}$, it is obtained from $\tilde{\mu}^{2i}$ by adding the unit vector e_1 if and only if i is a fixed point of $\mathcal{P}(\tilde{\mathcal{A}})$. Equivalently, this is the case if and only if $r+1-i$ is a fixed point of $\mathcal{P}(\mathcal{A})$, as can be seen by inspecting the evacuation diagram. □

6.5. Promotion and evacuation of oscillating tableaux. We now deduce Theorem 3.2 and Theorem 3.3 from the results in the preceding section, by demonstrating that oscillating tableaux can be regarded as special alternating tableaux.

For two partitions λ, μ we define

$$\lambda \vee \mu := \max(\lambda, \mu)$$
$$\lambda \wedge \mu := \min(\lambda, \mu)$$

where max and min are defined componentwise.

Consider an n-symplectic oscillating tableau $\mathcal{O} = (\omega^0, \omega^1, \dots, \omega^r)$. Then

$$\mathcal{A}_{\mathcal{O}} = [\omega^0, \omega^0]_n, [\omega^0 \vee \omega^1, \omega^0 \wedge \omega^1]_n, [\omega^1, \omega^1]_n, \dots, [\omega^{r-1} \vee \omega^r, \omega^{r-1} \wedge \omega^r]_n, [\omega^r, \omega^r]_n$$

is a GL(n)-alternating tableau. Because ω^i and ω^{i+1} differ by a unit vector, the staircase $[\omega^i \vee \omega^{i+1}, \omega^i \wedge \omega^{i+1}]_n$ is obtained by taking the larger partition as positive part, and the smaller partition as negative part.

If $\mathcal{O}$ is an oscillating tableau, the filling of $\mathcal{G}(\mathcal{A}_{\mathcal{O}})$ is symmetric with respect to the diagonal from the top-left to the bottom-right. In particular, if $\mathcal{O}$ has empty shape the filling is precisely the permutation obtained by interpreting the perfect matching as a fixed point free involution.

Conversely, suppose that $\mathcal{A}$ is an alternating tableau such that the filling of $\mathcal{G}(\mathcal{A})$ is symmetric with respect to the diagonal from the top-left to the bottom-right, and has no crosses on this diagonal. Then, taking the positive part of every second staircase in $\mathcal{A}$ we obtain an oscillating tableau $\mathcal{O}_{\mathcal{A}}$. The filling of $\mathcal{G}(\mathcal{O}_{\mathcal{A}})$ is precisely the part of $\mathcal{G}(\mathcal{A})$ below and to the left of the diagonal.

It is easy to see that the rotation of a fixed point free involution corresponds to the rotation of the associated perfect matching. Also, the reversal of the complement of a symmetric filling corresponds to the reversal of the associated perfect matching. Thus, it remains to show that this correspondence between oscillating tableaux and certain

alternating tableaux intertwines promotion of oscillating tableaux and alternating tableaux: $\mathcal{O}_{\mathrm{pr}\,\mathcal{A}} = \mathrm{pr}\,\mathcal{O}_{\mathcal{A}}$.

LEMMA 6.26. *The promotion of an oscillating tableau equals the oscillating tableau corresponding to the promotion of the associated alternating tableau:* $\mathcal{O}_{\mathrm{pr}\,\mathcal{A}} = \mathrm{pr}\,\mathcal{O}_{\mathcal{A}}$.

Proof. Let $\mathcal{O} = (\varnothing = \omega^0, \ldots, \omega^r = \varnothing)$ be an oscillating tableau. By Theorem 4.17 its promotion $\mathrm{pr}\,\mathcal{O} = (\varnothing = \acute{\omega}^0, \ldots, \acute{\omega}^r = \varnothing)$ can be computed using the local rule from Definition 4.12:

$$\acute{\omega}^{i-1} = \mathrm{dom}_{\mathfrak{H}_n}(\acute{\omega}^{i-2} + \omega^i - \omega^{i-1}), \tag{8}$$

where $\mathfrak{H}_n$ is the hyperoctahedral group, the Weyl group of the symplectic group $\mathrm{Sp}(2n)$. Recall that in this case the dominant representative of a vector is obtained by sorting the absolute values of its components into decreasing order.

Let $\mathcal{A}_{\mathcal{O}} = (\varnothing = \mu^0, \ldots, \mu^{2r} = \varnothing)$ be the alternating tableau associated with the oscillating tableau $\mathcal{O}$. Let $\acute{\mathrm{p}}\mathrm{r}\,\mathcal{A}_{\mathcal{O}} = (\varnothing = \acute{\mu}^0, \acute{\mu}^1, \ldots, \acute{\mu}^{2r-1}, \acute{\mu}^{2r} = \varnothing)$ be as in Definition 6.13 and let $\mathrm{pr}\,\mathcal{A}_{\mathcal{O}} = (\varnothing = \hat{\mu}^0, \ldots, \hat{\mu}^{2r} = \varnothing)$ be the promotion of $\mathcal{A}_{\mathcal{O}}$.

We have to show that for every square in the promotion diagram of the alternating tableau

$$\begin{array}{ccc} \mu^{2i-2} & \mu^{2i-1} & \mu^{2i} \\ \acute{\mu}^{2i-3} & \acute{\mu}^{2i-2} & \acute{\mu}^{2i-1} \\ \hat{\mu}^{2i-4} & \hat{\mu}^{2i-3} & \hat{\mu}^{2i-2} \end{array} \tag{9}$$

the positive parts of the four corners $\omega^{i-1} = \mu_+^{2i-2}$, $\omega^i = \mu_+^{2i}$, $\acute{\omega}^{i-2} = \hat{\mu}_+^{2i-4}$ and $\acute{\omega}^{i-1} = \hat{\mu}_+^{2i-2}$ satisfy Equation (8). Note that the positive parts and the negative parts of these staircases coincide. To avoid superscripts, we set $\mu^{2i-2} = [\lambda, \lambda]_n$, $\mu^{2i} = [\nu, \nu]_n$, $\hat{\mu}^{2i-4} = [\kappa, \kappa]_n$ and $\hat{\mu}^{2i-2} = [\mu, \mu]_n$.

Because the filling of $\mathcal{G}(\mathcal{A}_{\mathcal{O}})$ is symmetric, the position ℓ of the cross in the first row equals the position k of the cross in the first column. Thus, we have $\ell = k$ in Lemma 6.18. Let us consider the case $i \neq \ell$ first. We assume that $i < \ell$, the case of $i > \ell$ is very similar. If $i < \ell$, that is, $2i \leqslant 2\ell - 2$, the positive parts of staircases in the same column of the first two rows of diagram (9) coincide by Lemma 6.18(a). By Lemma 6.18(a′), the negative parts of staircases in the same column of the second two rows coincide.

Moreover, by construction of $\mathcal{A}_{\mathcal{O}}$, the staircase in the middle of the first row either equals $[\lambda, \nu]_n$ or $[\nu, \lambda]_n$. Let us assume the latter, the former case is dealt with similarly. For the staircase in the middle we then obtain, applying the local rule to the staircases on the top-left,

$$\acute{\mu}^{2i-2} = \mathrm{dom}_{\mathfrak{S}_n}\left([\lambda, \kappa]_n + [\nu, \lambda]_n - [\lambda, \lambda]_n\right) = [\nu, \kappa]_n.$$

Similarly, applying the local rule to the staircases on the bottom-right, we find

$$[\mu, \mu]_n = \mathrm{dom}_{\mathfrak{S}_n}\left([\hat{\mu}_+^{2i-3}, \kappa]_n + [\nu, \mu]_n - [\nu, \kappa]_n\right) = [\hat{\mu}_+^{2i-3}, \mu]_n.$$

Therefore, the square of staircases in diagram (9) has the following form:

$$\begin{array}{ccc} [\lambda,\lambda]_n & [\nu,\lambda]_n & [\nu,\nu]_n \\ [\lambda,\kappa]_n & [\nu,\kappa]_n & [\nu,\mu]_n \\ [\kappa,\kappa]_n & [\mu,\kappa]_n & [\mu,\mu]_n \end{array} \tag{10}$$

Because the negative parts of the four staircases in the lower left corner are all the same, the positive parts satisfy $\mu = \mathrm{dom}_{\mathfrak{S}_n}(\kappa+\nu-\lambda)$, and therefore also Equation (8).

It remains to show that Equation (8) also holds for $i = \ell$. By Lemma 6.18(c), the positive parts of the staircases $\mu^{2\ell-2}$, $\mu^{2\ell-1}$ and $\acute{\mu}^{2\ell-3}$ all coincide, and thus equal λ. Moreover, the positive part α of the staircase in the middle is obtained by adding a cell to the first column of the Ferrers diagram of λ.

By Lemma 6.18(c′), the positive parts of the staircases $\acute{\mu}^{2\ell-1}$, $\hat{\mu}^{2\ell-3}$ and $\hat{\mu}^{2\ell-2}$ all coincide, and thus equal μ. Moreover, the positive part α of the staircase in the middle is obtained by adding a cell to the first column of the Ferrers diagram of μ. Therefore $\lambda = \mu$.

Lemma 6.18(b) implies that the negative parts of the staircases $\mu^{2\ell-1}$, $\mu^{2\ell}$, $\acute{\mu}^{2\ell-2}$ and $\acute{\mu}^{2\ell-1}$ are all equal to ν. Finally, Lemma 6.18(a′) shows that the negative parts of the staircases $\acute{\mu}^{2\ell-3}$, $\acute{\mu}^{2\ell-2}$, $\hat{\mu}^{2\ell-4}$ and $\hat{\mu}^{2\ell-3}$ are all equal to κ. Thus $\nu = \kappa$ and diagram (9) has the form

$$\begin{array}{ccc} [\lambda,\lambda]_n & [\lambda,\nu]_n & [\nu,\nu]_n \\ [\lambda,\nu]_n & [\alpha,\nu]_n & [\lambda,\nu]_n \\ [\nu,\nu]_n & [\lambda,\nu]_n & [\lambda,\lambda]_n \end{array} \tag{11}$$

Considering the growth diagram $\mathcal{G}(\mathcal{A}_{\mathcal{O}})$, we additionally find that λ is obtained from ν by adding a cell to the first column. Thus, the vector $\nu + \nu - \lambda$ is obtained from ν by subtracting 1 from the entry at position $\ell(\lambda)$, which is 0 in ν. Taking the absolute values of the entries of the vector $\nu + \nu - \lambda$ then yields λ. □

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